Theoretical Computer Science 1
Exercise Sheet 1

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Hand in your solutions to the Vips directory of the StudIP course until Thursday, November 9th 2023 23:59. You should provide your solutions either directly as .pdf file or as a readable scan/photo of your handwritten notes. Submit your results as a group of four and state **all** members of your group with **student id, name and course**.

Homework Exercise 1: Lattices [7 points]

Let $(\mathbb{N}, \preceq)$ be a lattice, where $\preceq$ is a binary relation over $\mathbb{N}$ defined as follows: For $x, y \in \mathbb{N}$ the pair $x \preceq y$ holds if and only if $x = 0$ or $y = 1$ or $x = y \in \mathbb{N} \setminus \{0, 1\}$.

a) [1 point] Draw a Hasse-diagram of $(\mathbb{N}, \preceq)$ for the numbers up to 9.

b) [1 point] State $\top$ und $\bot$ of this lattice.

c) [3 points] State the values of the following joins and meets:

$$\bot \sqcup \top \qquad \bot \sqcap \top \qquad \top \sqcup 5 \qquad 6 \sqcap 7 \qquad \bot \sqcup 4 \qquad \bigsqcup\{n \in \mathbb{N} \mid n \text{ is odd}\}$$

d) [1 point] Is the height of this lattice finite? Is it bounded?

e) [1 point] Give a Hasse-diagram for a lattice which has finite but non-bounded height.

Homework Exercise 2: Some Lattice [13 points]

Let $M \subseteq \mathbb{N}$ be a finite, non-empty set and $M' := \{ \langle a, b \rangle \mid a, b \in M \text{ and } a < b \} \cup \{\square\}$ the set of ascending-sorted pairs from M, with an extra element $\square$.
Let $\preceq$ be a relation on M', defined as follows:

$$x \preceq y \quad \text{iff} \quad x = \square \quad \text{or} \quad (x = \langle a, b \rangle \text{ and } y = \langle c, d \rangle \text{ and } c \leq a \text{ and } b \leq d)\ .$$

a) [1 point] Draw a Hasse-diagram of $\langle M', \preceq \rangle$ with $M = \{0, 1, 2, 3, 4\}$.

In the following, let M again be a finite, non-empty set.

b) [5 points] Show that $\preceq$ is reflexive, transitive and antisymmetrical.
By definition, $\langle M', \preceq \rangle$ is then a partial order.

c) [4 points] Show that the join $\bigsqcup X$ and the meet $\bigsqcap X$ exist for each subset $X \subseteq M'$.
By definition, $\langle M', \preceq \rangle$ is then a finite complete lattice.

d) [1 point] State $\top, \bot$ for $\langle M', \preceq \rangle$, depending on M.

e) [2 points] Does $\langle M', \preceq \rangle$ stay complete, if $M \subseteq \mathbb{N}$ is infinite?

Exercise 3:
Let $M_1 \subseteq \mathbb{N}$ and $M_2 \subseteq \mathbb{N}$ be two finite sets and $M = M_1 \times M_2$ the set of all pairs (a, b) with $a \in M_1$ and $b \in M_2$. Let $\preceq$ be a relation on M, defined as follows:

$$\langle a_1, b_1 \rangle \preceq \langle a_2, b_2 \rangle \quad \text{gdw.} \quad a_1 \geqslant a_2 \text{ und } b_1 \geqslant b_2$$

where $\leqslant$ is the common "less or equals" relation on natural numbers.

Show that $\langle M, \preceq \rangle$ is then a complete lattice.

Does $\langle M, \preceq \rangle$ stay complete, if $M_1 \subseteq \mathbb{N}$ is infinite?

Exercise 4:
Let $\langle D_1, \preceq_1 \rangle$ and $\langle D_2, \preceq_2 \rangle$ be complete lattices. The **product lattice** is defined as $\langle D_1 \times D_2, \preceq \rangle$, where $\preceq$ is the **product ordering** on tuples with $\langle d_1, d_2 \rangle \preceq \langle d_1', d_2' \rangle$ if and only if $d_1 \preceq_1 d_1'$ and $d_2 \preceq_2 d_2'$.

Show that the product lattice is indeed a complete lattice.

Prove the following; The product lattice $\langle D_1 \times D_2, \preceq \rangle$ satisfies ACC if and only if $\langle D_1, \preceq_1 \rangle$ and $\langle D_2, \preceq_2 \rangle$ both satisfy ACC.

Exercise 5:
Let $\langle D, \leqslant \rangle$ be a lattice and $x, y \in D$ be two arbitrary elements.

Show that if $f : D \to D$ is monotone, then $f(x \sqcup y) \geqslant f(x) \sqcup f(y)$ holds.

$f : D \to D$ is called **distributive**, if $f(x \sqcup y) = f(x) \sqcup f(y)$ for all $x, y \in D$.
Show that if f is distributive then f is also monotone.

Exercise 6:
Let $\langle D, \sqsubseteq \rangle$ be a lattice. Prove the first two statements from lemma 1.8 from the lecture: If $\bigsqcap D$ is defined, then the identity $\bigsqcap D = \bigsqcup \varnothing$ holds. Analoguously $\bigsqcup D = \bigsqcap \varnothing$, if $\bigsqcup D$ is defined.

Exercise 7:
Show the last statement of lemma 1.8: All finite lattices are complete.

Exercise 8:
Let M be a set. Show that $\langle \mathcal{P}(M), \subseteq \rangle$ is a complete lattice.