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Exercises to the lecture
Semantics
Sheet 5

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Exercise 5.1 (Böhm trees)
For the following $\lambda\mathsf{Y}$ terms over the signature

$$\Sigma = \{\texttt{if} : o \to o \to o \to o, \texttt{or}, +, - : o \to o \to o, \texttt{iszero}, \texttt{a} : o \to o\},$$

create their Böhm trees up to depth 4, note their type and describe what they compute in the data domain $\mathbb{N}$. Interprete Y as the greatest fixed point.

- $\mathsf{Y}\lambda x^o.x^o$
- $\mathsf{Y}\lambda x^o.\texttt{a}x^o$
- $\mathsf{Y}\lambda x^o. + x^o\, y^o$
- $\mathsf{Y}\lambda F^{o\to o}\lambda a^o.$
  $\texttt{if}\,(\texttt{or}\,(\texttt{iszero}\,(-\,a^o\,1))\,(\texttt{iszero}\,a^o))$
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  $(+\,(F^{o\to o}\,(-\,a^o\,1)\,(F^{o\to o}\,(-\,a^o\,2))))$

Exercise 5.2 (GFP-models for $\lambda\mathsf{Y}$ terms)
Proof the following Lemma from the lecture:
If $M =_{\beta\delta} N$, then for all GFP-models S and variable assignments ν, $[\![M]\!]^\nu_S = [\![N]\!]^\nu_S$.

Exercise 5.3 (TAC automata)
The definition for trees was $t : \{1,2\}^* \to \Sigma \cup \{\Omega\}$, such that

1. If $uv \in \mathrm{dom}(t)$ then $u \in \mathrm{dom}(t)$
2. If $u \in \mathrm{dom}(t)$ and $t(u) \in \Sigma_2$, then $u.1, u.2 \in \mathrm{dom}(t)$
3. If $u \in \mathrm{dom}(t)$ and $t(u) \in \Sigma_0 \cup \{\Omega\}$, then u is leaf.

A top-down tree automaton has the same syntactical definition as insightful TAC automata. However, its language is only defined for finite trees. When we interpret a given top-down tree automaton, we can find a pumping lemma. For this, we need to define for $u \in \mathrm{dom}(t)$:

- $t_u :=$ the subtree rooted in u
- $t - u :=$ the tree t', such that $\mathrm{dom}(t') = \mathrm{dom}(t) - u.\{1,2\}^+$ and for each $u' \in \mathrm{dom}(t')$, $t'(u') = t(u')$.
- $t +_u t'$, where $t(u) = t'(\varepsilon)$ is defined by $t(v)$ for $v \in \mathrm{dom}(t - u)$ and by $t'(v)$ for $u.v$ with $v \in \mathrm{dom}(t')$.

- if $t(\varepsilon) = t(u)$, $t^{n,u} = \underbrace{t +_u (t +_u (t +_u (\dots)))}_{n \text{ times}}$ and $t^{*,u} = \bigcup_{n\in\mathbb{N}}\{t^{n,u}\}$

Lemma. *Given a top-down tree automaton with n states and a tree $t \in L(A)$ with depth $> n$, there are vertices $u, u.v \in \mathrm{dom}(t)$ such that $t = ((t - u) +_u ((t_u - v) +_v t_{uv}))$ and for each $n \in \mathbb{N}$, $((t - u) +_u ((t_u - v)^{n,v} +_{v^n} t_{uv})) \in L(A)$.*

Given a top-down tree automaton with n states accepting a language $L(A)$. What is the language of the same automaton interpreted as insightful TAC automaton?