Games with perfect information
Exercise sheet 10

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Out: June 6 Due: June 20

Submit your solutions on Wednesday, June 20, at the beginning of the lecture.
Please submit in groups of three persons.

Exercise 1: From parity to Muller

a) Let $G = (V_\square \uplus V_\bigcirc)$ be a finite, deadlock-free graph. Consider the parity game $\mathcal{G}^{\text{parity}}$ defined on G by some priority assignment $\Omega\colon V \to \{0, \ldots, n\}$.

   Present a Muller judgment $\mathit{judgment}\colon \mathcal{P}(V) \to \{\bigcirc, \square\}$ such that the corresponding Muller game $\mathcal{G}^{\text{Muller}}$ is equivalent to $\mathcal{G}^{\text{parity}}$: Any position $x \in V$ is winning for some player $☆$ in $\mathcal{G}^{\text{Muller}}$ if and only if it is winning for this player in $\mathcal{G}^{\text{parity}}$.

b) We call a Muller game $\mathcal{G}^{\text{Muller}}$ **union-closed** if its defining judgment has the following property: If $\mathit{judgment}(X) = ☆$ and $\mathit{judgment}(Y) = ☆$ for some sets $X, Y \subseteq V$, then $\mathit{judgment}(X \cup Y) = ☆$.

   Check that the game you have constructed in Part a) is union-closed.

Note: One can show that if a Muller game is union-closed, and $x \in V$ is winning for some player $☆$, then $☆$ has a positional winning strategy from x.

Exercise 2: Gale-Stewart games as graph games

Let $\mathcal{G}(A, B)$ be a Gale-Stewart game. Define an equivalent game over a graph with set of positions

a) $V = A \times \{\bigcirc, \square\}$,

b) $V = A^*$.

In each case, specify the ownership, the arcs, the winning condition and the initial position of interest.

Exercise 3: Reachability games as Gale-Stewart games
Let $\mathcal{G}$ be a reachability game, specified as usual by a game arena $G = (V_\square \uplus V_\bigcirc, R)$ and a winning set $V_{reach} \subseteq V$. For simplicity, let us assume that G is deadlock-free and bipartite: Any move from some position $x \in V_\bigcirc$ leads to a position $y \in V_\square$ and vice versa, i.e. the players take turns alternately. Furthermore, we fix the initial position $x_0 \in V_\bigcirc$.

Design a Gale-Steward game $\mathcal{G}(V, B)$ where the actions are nodes of G and B is such that

1. the existential player loses if she does not start in x_0,
2. the existential player loses if she picks an illegal move, i.e. if the play p is of the shape $p = p'.x_i.x_{i+1}.p''$ where i is odd and $(x_i, x_{i+1}) \notin R$,
3. the universal player loses if she picks an illegal move, i.e. if the play p is of the shape $p = p'.x_i.x_{i+1}.p''$ where i is even and $(x_i, x_{i+1}) \notin R$,
4. any play that does not fall into one of the Cases 1. to 3. is won by the existential player if and only if it contains a position from V_{reach}.

Argue briefly that your set B enforces the desired behavior.

Note: If a play falls into several cases, i.e. into 2. and 3. if both players cheat, you may resolve this as you wish.

Exercise 4: Cardinality and functions
Recall that a function $f: X \to Y$ is called **injective** (or an **injection**) if for $x \neq x'$ we have $f(x) \neq f(x')$. A function is called **surjective** (or a **surjection**) if for any $y \in Y$, there is some $x \in X$ with $f(x) = y$. It is called **bijective** (or a **bijection**) if it is both injective and surjective.

In the rest of this exercise, assume that $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ are finite sets.

a) Prove that if there is an injection $f: X \to Y$ if and only if $|X| \leq |Y|$.

b) Prove that if there is a surjection $f: X \to Y$ if and only if $|X| \geq |Y|$.

c) Prove that there is an injection $g: Y \to X$ if and only if there is a surjection $f: X \to Y$.

d) Prove that if there is an injection $f_i: X \to Y$ and a surjection $f_s: X \to Y$, then there is a bijection $f_b: X \to Y$.

Note: This exercise is a preparation for the lecture on June 20.

Note: The properties that you have proven in Part c) and d) also hold for infinite sets. However, their proof is much more complicated in this case and involves the axiom of choice.