Games with perfect information
Exercise sheet 7

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Out: May 9 Due: May 30

This is the exercise sheet for the lecture that will be given on May 16.

Submit your solutions on Wednesday, May 30, at the beginning of the lecture.
Please submit in groups of three persons.

Exercise 1: Parity tree automata

Consider the ranked alphabet $\Sigma = \{a_{/2}, b_{/2}\}$. Note that Σ-trees are so-called *full* infinite binary trees.

a) Consider the PTA $A_1 = (\Sigma, \{q_0, q_1\}, q_0, \rightarrow, \Omega)$ with

$$\begin{aligned} \rightarrow_a &= \{(q_0, (q_1, q_1))\} , \\ \rightarrow_b &= \{(q_1, (q_0, q_0))\} , \\ \Omega(q_0) &= \Omega(q_1) = 0 . \end{aligned}$$

Describe its language $\mathcal{L}(A_1)$.

b) Consider the PTA $A_2 = (\Sigma, \{q_+, q_-\}, q_+, \rightarrow, \Omega)$ with

$$\begin{aligned} &\rightarrow_a = \{(q_+, (q_-, q_-))\} , \\ &\rightarrow_b = \{(q_+, (q_+, q_-)),\ (q_+, (q_-, q_+)),\ (q_-, (q_-, q_-))\} , \\ \Omega(q_-) = 0 , \qquad &\Omega(q_+) = 1 . \end{aligned}$$

Formally prove that $\mathcal{L}(A_2)$ is exactly the set of Σ-trees in which exactly one single node is labeled by a.

Remark: A_2 is non-deterministic, and one can prove that there is no deterministic PTA A accepting the same language.

c) Present a PTA A_3 whose language is the set of Σ-trees in which exactly one branch contains infinitely many nodes labeled by a.

Argue that your automaton indeed has this property.

Exercise 2: Closure properties of regular languages of infinite trees

Prove that regular languages of infinite trees are closed under union, intersection, and projection.

Let $A = (\Sigma, Q, q_0, \rightarrow, \Omega), A' = (\Sigma, Q', q_0', \rightarrow', \Omega')$ be PTAs over the same ranked alphabet Σ.

a) Show how to construct a PTA $A_\cup$ with $\mathcal{L}(A_\cup) = \mathcal{L}(A) \cup \mathcal{L}(A')$.

b) Show how to construct a PTA $A_\cap$ with $\mathcal{L}(A_\cap) = \mathcal{L}(A) \cap \mathcal{L}(A')$.

   Hint: Assume that you had already proven Rabin's tree theorem.

c) Let Σ' be a ranked alphabet, and $f: \Sigma \rightarrow \Sigma'$ be a **rank preserving function**, i.e. we have $\text{rank}_\Sigma(a) = \text{rank}_{\Sigma'}(f(a))$ for all $a \in \Sigma$. For a Σ-tree $\mathcal{T}$, we define $f(\mathcal{T})$ to be the Σ'-tree in which the label a of each node is replaced by $f(a)$. Note that the fact that f is rank-preserving is crucial for $f(\mathcal{T})$ being a Σ'-tree.

   For a language of Σ-trees $\mathcal{L}$, we define $f(\mathcal{L}) = \{f(\mathcal{T}) \mid \mathcal{T} \in \mathcal{L}\}$. Show how to construct a PTA $A_f = (\Sigma', Q_f, q_{0f}, \rightarrow_f, \Omega_f)$ with $\mathcal{L}(A_f) = f(\mathcal{L}(A))$.

Exercise 3: Correctness of the tree acceptance game

Let $\mathcal{T}$ be a Σ-tree and let A be a PTA. Consider the parity game $\mathcal{G}(\mathcal{T}, A)$ as defined in Definition 8.12 of the lecture notes, and assume that it is deadlock-free.

a) Assume that the existential player has a positional winning strategy $s_\bigcirc$ from position (ε, q_0) in $\mathcal{G}(\mathcal{T}, A)$. Present an accepting run of $\mathcal{T}$ on A.

   Hint: Construct the run inductively, guided by $s_\bigcirc$.

b) Assume that the universal player has a positional winning strategy $s_\Box$ from position (ε, q_0) in $\mathcal{G}(\mathcal{T}, A)$. For each candidate run of A on $\mathcal{T}$, identify a branch on which the acceptance condition is violated.

Exercise 4: Deadlocks in the tree acceptance games

Let $\mathcal{T}$ be a Σ-tree and let A be a PTA. Consider the parity game $\mathcal{G}(\mathcal{T}, A)$ as defined in Definition 8.12 of the lecture notes.

The game arena of $\mathcal{G}(\mathcal{T}, A)$ is not necessarily deadlock-free.

a) In which case can deadlocks occur?

b) Modify the game arena such that it becomes deadlock free such that the correctness of the tree acceptance game (Lemma 8.13 in the lecture notes) is preserved.

c) How can one modify the automaton A without changing its language such that $\mathcal{G}(\mathcal{T}, A)$ is deadlock-free without modification?