Games with perfect information
Exercise sheet 6

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Out: May 9 Due: May 16

Submit your solutions on Wednesday, May 16, at the beginning of the lecture.
Please submit in groups of three persons.

Exercise 1: Is it a trap?

a) Formally prove Part a) of Lemma 6.9 from the lecture notes:

Let $Y \subseteq V$ and $☆ \in \{\bigcirc, \square\}$. The complement of the attractor $V \setminus \mathrm{Attr}_{☆}(Y)$ is a trap for player ☆.

b) Construct a game arena and a set Y such that $\mathrm{Attr}_{☆}(Y)$ is not a trap for any of the players. Proof that these properties hold.

Exercise 2: It's a trap!

Formally prove Lemma 6.12 from the lecture notes:

Let $X \subseteq V$ be a trap for player ☆ in $\mathcal{G}$ and let $s_{\overline{☆}}$ be a strategy for the opponent $\overline{☆}$ that is winning from some vertex $x \in X$ in the subgame $\mathcal{G}_{\restriction X}$. Then $s_{\overline{☆}}$ is also winning from x in the game $\mathcal{G}$.

Exercise 3: Weak parity games

A **weak parity game** is given by a game arena $G = (V_\square \uplus V_\bigcirc, R)$ and a priority function Ω. Instead of considering the highest priority that *occurs infinitely often* to determine the winner of a play, we consider the highest priority that *occurs at all*.

Formally, for an infinite sequence $p \subseteq A^\omega$, we define the **occurrence set**

$$\mathrm{Occ}(p) = \{a \in A \mid \exists i \in \mathbb{N} \colon p_i = a\} .$$

The winner of the weak parity game given by G and Ω is determined by the **weak parity winning condition**:

$$\begin{aligned} win \;:\; Plays_{max} &\to \{\bigcirc, \square\} \\ p &\mapsto \begin{cases} \bigcirc, & \text{if } \max \mathrm{Occ}(\Omega(p)) \text{ is even}, \\ \square, & \text{else, i.e. if } \max \mathrm{Occ}(\Omega(p)) \text{ is odd}. \end{cases} \end{aligned}$$

a) Present an algorithm that, given a weak parity game on a finite, deadlock-free game arena, computes the winning regions of both players. Briefly argue that your algorithm is correct.

Hint: Attractors!

b) Is the winning condition of weak parity games prefix-independent, i.e. does Lemma **??** hold?

Do uniform positional winning strategies exist?

Algorithm: Zielonka's recursive algorithm

Input: parity game $\mathcal{G}$ given by $G = (V_\Box, V_\bigcirc, R)$ and Ω.

Output: winning regions $W_\Box$ and $W_\bigcirc$.

Procedure `solve`($\mathcal{G}$)

$n = \max_{x \in V} \Omega(x)$
if $n = 0$ **then**
  return $W_\bigcirc = V, W_\Box = \varnothing$
else
  $N = \{x \in V \mid \Omega(x) = n\}$
  if n even **then**
    $\star = \bigcirc, \overline{\star} = \Box$
  else
    $\star = \Box, \overline{\star} = \bigcirc$
  end if
  $A = \mathrm{Attr}^{\mathcal{G}}_{\star}(N)$
  $W'_\bigcirc, W'_\Box = solve(\mathcal{G}_{\restriction V \setminus A})$
  if $W'_\star = V \setminus A$ **then**
    return $W_\star = V, W_{\overline{\star}} = \varnothing$
  else
    $B = \mathrm{Attr}^{\mathcal{G}}_{\overline{\star}}(W'_{\overline{\star}})$
    $W''_\Box, W''_\bigcirc = solve(\mathcal{G}_{\restriction V \setminus B})$
    return $W_\star = W''_\star, W_{\overline{\star}} = W''_{\overline{\star}} \cup B$
  end if
end if

Exercise 4: Algorithmics of parity games

Use Zielonka's recursive algorithm to solve the following parity game.