Games with perfect information
Exercise sheet 3

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Out: April 19 — Due: April 24

Submit your solutions until Monday, April 24, 14:00, in the box next to office 343.

Exercise 1: Determinacy of games of finite length

When considering chess, we have already used the theorem that you will prove in this exercise.

Let $\mathcal{G} = (G, win)$ be a game such that each maximal play of $\mathcal{G}$ has finite length. Then $\mathcal{G}$ is determined, i.e. every position is winning for exactly one of the players, $V = W_{\bigcirc} \uplus W_{\square}$.

Hint: Construct a reachability game whose set of positions is $Plays_{\mathcal{G}}$.

Exercise 2: Attractors have attractive algorithmics!

a) Prove that if $\text{Attr}^i_{\star}(B) = \text{Attr}^{i+1}_{\star}(B)$, then we have $\text{Attr}^i_{\star}(B) = \text{Attr}_{\star}(B)$.

   Conclude that if the set of positions V is finite, we have $\text{Attr}_{\star}(B) = \text{Attr}^{|V|}_{\star}(B)$.

b) Let $G = (V, E)$ be a finite game arena, and let $B \subseteq V$ be a set. We consider the reachability game on G with respect to B. As in the lecture, we assume that refuter wants to reach B, while prover wants to prevent this.

   Write down pseudo-code for an algorithm that computes the winning region $W_{\bigcirc}$ of refuter, and at the same time computes uniform positional winning strategies $s_{\bigcirc}, s_{\square}$ for both players.

c) Consider a 2×2-variant of tic tac toe, i.e. tic tac toe played on a 2×2 matrix. We assume that $\bigcirc$ starts. The player that is first able to put 2 of her marks into one row, column or diagonal wins, and the game then stops.

   Formalize this game as a reachability game and solve it using the attractor algorithm.

Exercise 3: Graphs with infinite out-degree

In the lecture, we made the assumption that the out-degree of the graph is finite. In this exercise, we want to understand this restriction.

Let $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ denote the positive natural numbers. We consider the graph $G = (V, R)$ given by

$$V = \{start, goal\} \cup \bigcup_{i \in \mathbb{N}^+} Path_i\,, \text{ where for each } i \in \mathbb{N}^+, \text{ we have } Path_i = \left\{p^i_1, p^i_2, \ldots, p^i_i\right\},$$

$$R = \bigcup_{i \in \mathbb{N}^+} \left\{\left(start, p^i_1\right)\right\} \cup \bigcup_{i \in \mathbb{N}^+} \left\{\left(p^i_i, goal\right)\right\} \cup \bigcup_{i \in \mathbb{N}^+} \bigcup_{j=1}^{i-1} \left\{\left(p^i_j, p^i_{j+1}\right)\right\}.$$

We want to consider a reachability game on G with respect to the winning set $\{goal\}$, i.e. refuter ○ needs to reach the position *goal*, prover wants to prevent this.

a) Draw a schematic representation of the graph G, e.g. involving the vertices $\{start, goal\}$ and the positions in $Path_i$ for $i \leq 4$.

b) Assume that all positions are owned by refuter. For each position $x \in V$, give the minimal $i_x \in \mathbb{N}$ such that $x \in \text{Attr}^{i_x}_{\bigcirc}(\{goal\})$, respectively $i_x = \infty$ if no such i_x exists.

   Present a winning strategy for the reachability game from the position *start*.

c) Assume that all positions are owned by prover. For each position $x \in V$, give the minimal i_x such that $x \in \text{Attr}^{i_x}_{\bigcirc}(\{goal\})$, respectively $i_x = \infty$ if no such i_x exists.

   Which player wins the reachability game from *start*?