Games with perfect information
Exercise sheet 2

Sebastian Muskalla
Prof. Dr. Roland Meyer
TU Braunschweig
Summer term 2017

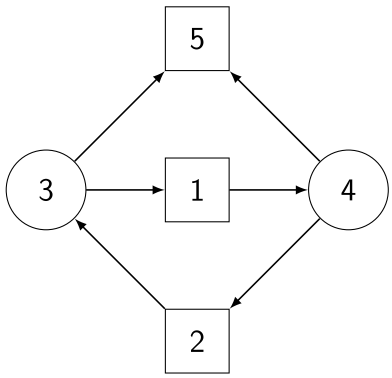
Out: April 12, **Updated: April 15** Due: April 18

You can submit your solutions on Tuesday, April 18, at the beginning of the exercise classes (since Monday, April 17, is a public holiday.) We will grade and return the submissions as soon as possible! Happy Easter!

Exercise 1: Positional and uniform strategies

Note: Look up the definition of *positional* strategies in the lecture notes before doing this exercise.

If a game arena has finitely many positions, we can explicitly give it as a graph. For this exercise, we consider a game on the following game arena $G = (V, R)$. Positions owned by prover $\Box$ are drawn as boxes, positions owned by refuter $\bigcirc$ are drawn as circles. The numbers should denote the names of the vertices, i.e. $V = \{1, \ldots, 5\}$.



We consider the following winning condition: A maximal play is won by refuter if and only if the positions 3, 4 and 5 are each visited exactly once. Otherwise, prover wins the play.

a) What are the winning regions of refuter and prover? Present a single strategy $s_{\bigcirc} \colon Plays_{\bigcirc} \to V$ that is winning from all positions x in the winning region $W_{\bigcirc}$ of refuter. Argue shortly why your strategy is indeed winning from these positions.

   Note: Such a strategy is called a *uniform* winning strategy.

b) For each vertex $x \in W_{\bigcirc}$ in the winning region of refuter, present a positional strategy for refuter $s_{\bigcirc,x} \colon \{3, 4\} \to V$ such that $s_{\bigcirc,x}$ is winning from x.

c) Prove that there is no uniform positional winning strategy for refuter, i.e. no single positional strategy that wins from all $x \in W_{\bigcirc}$.

Exercise 2: Tic-tac-toe

Consider the popular game **tic-tac-toe**, see e.g. `https://en.wikipedia.org/wiki/Tic-tac-toe`.

Formalize the game, i.e. formally define a game $\mathcal{G} = (G, win)$ consisting of a game arena and a winning condition that imitates the behavior of tic-tac-toe.

Assume that player $\bigcirc$ makes the first mark, and the other player wins in the case of a draw.

Exercise 3: Language inclusion as a game

Note: You may need to recall the definitions of finite automata for this exercise.

Consider two non-deterministic finite automata (*NFAs*) $A = (Q_A, q_{0A}, \rightarrow_A, Q_{FA})$, and $B = (Q_B, q_{0B}, \rightarrow_B, Q_{FB})$ over the same alphabet of input symbols Σ. We want to construct a game that is won by prover $\square$ if and only if the regular language accepted by A is included in the regular language accepted by B, i.e. $\mathcal{L}(A) \subseteq \mathcal{L}(B)$.

Our approach is to let each of the players control one of the automata. Refuter controls automaton A, and her goal is to disprove (or *refute*) inclusion. To do so, she step-by-step picks a run of A such that the corresponding word is accepted by A, but not accepted by B. Prover wants to prove inclusion and controls automaton B. She has to react to the moves made by refuter to find an accepting run of automaton B for the word chosen by refuter.

More precisely, the game works as follows:

- A configuration of the game consists of a state q_A resp. q_B of each automaton.
- The players alternately takes turns, starting with refuter $\bigcirc$.
- In each of her turns, refuter selects a transition $q_A \xrightarrow{a}_A q'_A$ of the automaton A.
- In the following turn of prover, she selects a transition $q_B \xrightarrow{a}_B q'_B$ of B. Note that is has to be labeled by the same letter $a \in \Sigma$ that was picked by refuter in the previous move.
- A maximal play of the game is won by refuter if it visits a configuration in which the state q_A of A is final, but the state q_B of B is not final (Intuitively, this means that the word chosen step-by-step by refuter is accepted by A, but not accepted by B.) It is also won by refuter if it ends in a position in which prover cannot react to a move made by refuter, i.e. there is no transition of B with the required letter. It is won by prover otherwise.

a) Formalize the game, i.e. formally define a game arena G and a winning condition *win* such that the game $\mathcal{G} = (G, win)$ has the behavior described above.

b) Let x be the configuration of the game consisting of the initial states q_{0A} and q_{0B} of both automata. We would like to have the following result:
"x is winning for prover if and only if the inclusion $\mathcal{L}(A) \subseteq \mathcal{L}(B)$ holds."
Prove that this is **not** true in general by considering the following automata over the alphabet $\{a, b, c\}$.