4. Regular Separability of VASS Reachability Languages

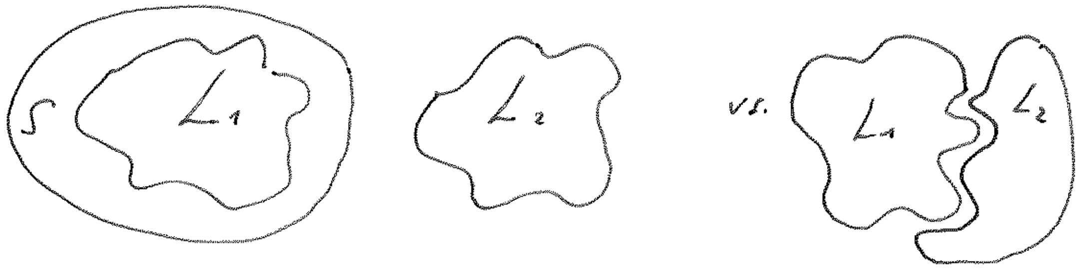
<u>Goal</u>: Prove that regular separability of VASS reachability languages is decidable.

<u>Definition</u>:

Languages $L_1, L_2 \subseteq \Sigma^*$ are <u>separable by a regular language</u>, denoted $L_1 | L_2$, if there is a regular language $S \subseteq \Sigma^*$ with

$$L_1 \subseteq S \quad \text{and} \quad S \cap L_2 = \emptyset.$$

In this case, S is called a <u>separator</u>.



We will study the regular separability problem for VASS languages and for $\mathbb{Z}$-VASS languages. Therefore, we give a definition parametrized in $\mathbb{X} \in \{\mathbb{N}, \mathbb{Z}\}$:

<u>$\mathbb{X}$ - REGSEP</u>:

<u>Given</u>: Initialized (with initial and final configuration) VASS V_1 and V_2 over Σ.

<u>Question</u>: Does $L_{\mathbb{X}}(V_1) \,|\, L_{\mathbb{X}}(V_2)$ hold?

The main result will be this.

<u>Theorem ([Keskin, M., LICS '24])</u>:

$\mathbb{N}$-REGSEP is decidable and F_ω-complete.

We can effectively compute a separator in this (time and space) bound.

For $\mathbb{Z}$-VASS languages, the result is known.

Theorem ([Clemente, Czerwiński, Lasota, Paperman, ICALP'17]):

$\mathbb{Z}$-REGSEP is decidable and
a separator can be computed
with elementary resources

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}}_{\text{for a fixed tower of exponents.}}$$

4.1 The Transducer Trick

(appeared in the above ICALP paper,
turned into an approach by [Czerwiński & Zetzsche, LICS'20])

Goal: Reduce $\mathbb{N}$-REGSEP to a problem
that only expects a single language as input
and fixes the second.

Definition:

- The Dyck language D_n (where n is the number of counters in the second VASS)
is defined over the alphabet

$$\Sigma_n := \{a_i, \overline{a_i} \mid i \in d_y\} \text{ with } d_y := [1,n]$$

- Intuitively:
  - a_i = increment counter i } Dyck visibility
  - $\overline{a_i}$ = decrement counter i }
  - keep counters non-negative
  - initially and finally zero.
- Formally, the Dyck language is accepted
by the following initialized VASS:

$(\mathcal{D}_n, (v, \bar{0}), (v, \bar{0}))$ with

$$\mathcal{D}_n := (\{v\}, \Sigma_n, \underset{\substack{= \\ d_y}}{[1,n]}, E) \text{ with}$$

$$E = \{ (v, a_i, e_i, v), (v, \bar{a}_i, -e_i, v) \mid i \in d_y \}$$

i-th unit vector

<u>Goal</u>: Express $\mathbb{N}$-VASS languages as transductions of $\mathcal{D}_n$.

<u>Definition</u>:

- A <u>rational</u> <u>transduction</u> between alphabets Σ and Γ is a relation

  $$T \subseteq \Sigma^* \times \Gamma^*$$

  that is accepted by a finite transducer.

- A finite transducer is a finite automaton

  $$A = (Q, q_0, \delta, Q_F)$$

  with $\delta \subseteq Q \times \Sigma^* \times \Gamma^* \times Q$.

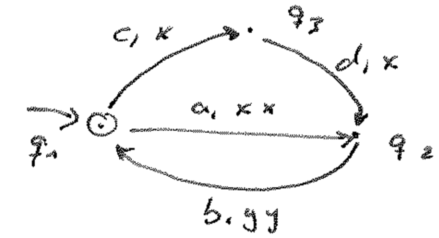
  Note that the transitions can be normalized to $(\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\})$.

- Alternatively, one can define rational transductions as homomorphic images of regular languages:

  $$T = h(R)$$

  with $h: \Delta^* \longrightarrow \Sigma^* \times \Gamma^*$.

Example:



Lemma:

- If $T \subseteq \Delta^* \times \Sigma^*$ and $S \subseteq \Sigma^* \times \Gamma^*$ are rational transductions, so is

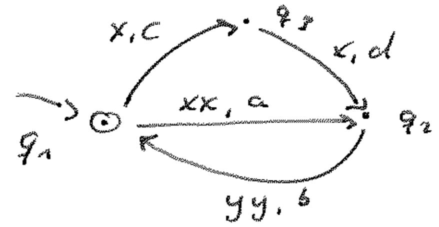
$$S \circ T := \{(u,w) \mid \exists v \in \Sigma^*.\ (u,v) \in T \wedge (v,w) \in S\}$$
$$\subseteq \Delta^* \times \Gamma^*.$$

- Also
$$T^{-1} := \{(w,u) \mid (u,w) \in T\} \subseteq \Sigma^* \times \Delta^*$$
is a rational transduction.

Proof: $\circ$: use a product construction

-1 : invert the labels.

Example:



Lemma:

Let V be an initialized $\mathbb{N}$-VASS over Σ with n-many counters.
There is a rational transduction $T_V \subseteq \Sigma_n^* \times \Sigma^*$ so that
$$L(V) = T_V(D_n).$$

Proof (idea):

- The Dyck language implements the counter operations.
- The finite transducer is the control of V.

Let $V = (W, (q_{in}, v_{in}), (q_{fin}, v_{fin}))$ with $W = (Q, \Sigma, C, \delta_W)$.

The finite transducer accepting T_V is

$$\mathcal{A}_V := (Q \uplus \{q_1, q_2\}, q_1, \delta_{\mathcal{A}}, \{q_2\})$$

with

$$\delta_{\mathcal{A}} := \{(q, word_{\bar{x}}, a, q') \mid q \xrightarrow{\bar{x}, a} q' \in \delta_W\}$$

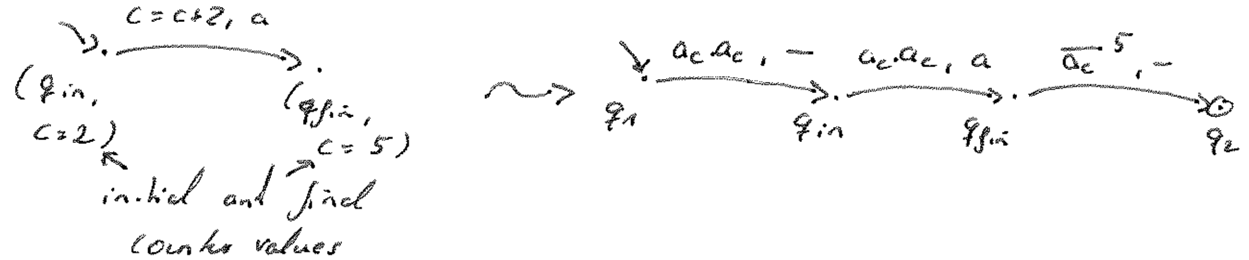
↳ $c_i{+}{+}$ yields a_i

$c_i{-}{-}$ yields $\bar{a}_i$

$$\cup \{(q_1, word_{v_{in}}, -, q_{in}), (q_{fin}, word_{v_{fin}}, -, q_2)\}.$$

establish initial and final counter valuations, recall that the Dyck language contains words that take the counters from zero to zero.

Example:



Lemma (Closure of VASS under rational transductions):
Let V be an initialized VASS over Σ and T a rational transduction from Σ to Γ. We can compute in elementary time a VASS V' over Γ so that

$$L(V') = T(L(V)).$$

Proof:
Use a product construction. This is not just polynomial due to the counter updates. □

Proposition (Transducer Trick [Clemente et al., ICALP '17], [Czerwiński & Zetzsche, LICS '20])

Consider $\mathbb{N}$-VASS V and W.

Let n be the number of counters in W.

Then

$$L(V) \mid L(W) \quad \text{iff} \quad L(V) \mid T_W(D_n)$$
$$\text{iff} \quad T_W^{-1}(L(V)) \mid D_n.$$

Proof:

"$\Rightarrow$" Assume S separates $L(V)$ and $L(W)$.

We show that $T_W^{-1}(S)$ separates $T_W^{-1}(L(V))$ and D_n.

Inclusion:

Since $L(V) \subseteq S$, we have

$$T_W^{-1}(L(V)) \subseteq T_W^{-1}(S).$$

Intersection emptiness:

Since $S \cap L(W) = \emptyset$, we have

$$T_W^{-1}(S) \cap D_n = \emptyset.$$

Otherwise, $x \in T_W^{-1}(S) \cap D_n$ yields

$$T_W(x) \cap S \cap \underbrace{T_W(D_n)}_{= L(W)} \neq \emptyset. \quad ↯ \; S \cap L(W) = \emptyset.$$

"$\Leftarrow$" If $T_W^{-1}(L(V)) \mid D_n$, then $D_n \mid T_W^{-1}(L(V))$.

Like in the first direction,

$$(T_W^{-1})^{-1}(D_n) \mid L(V) \quad \text{follows.}$$

Since $(T_W^{-1})^{-1} = T_W$ and $T_W(D_n) = L(W)$, we have

$$L(W) \mid L(V).$$

$\square$