# 3.4 Establishing Perfectness by Decomposition

**Recall:** An MGTS $W$ is perfect, if

for every precovering graph $G$ in $W$:

(i) $CS_{up}(G) \neq \emptyset \neq CS_{down}(G)$

(ii) $supp(Char(W))$ justifies the unboundedness in $G$:

a) $\forall c \in C. \quad G.\underset{c_{out}}{c_{in}}(c) = \omega \Rightarrow x[G, \underset{out}{in}, c] \in supp(Char(W))$

b) $\forall t \in G.T. \qquad x[t] \in supp(Char(W))$.

**Goal:** If $W$ is not perfect,

we can compute a finite set $\mathbb{W}$ of MGTS

so that

- $\forall W' \in \mathbb{W}. \quad W' < W$ with $<$ well-founded.
- $IAcc_{\mathbb{Z}}(\mathbb{W}) \subseteq IAcc_{\mathbb{Z}}(G)$
- $IAcc_{\mathbb{N}}(W) = IAcc_{\mathbb{N}}(G)$.

**Approach:** Consider each case in which perfectness may fail.

## 3.4.1 Case (i): $CS_{up}(G) = \emptyset$

**Idea:** If the precovering graph $G$ does not have an uppumping sequence:

(1) Build a coverability graph from $G.c_{in}$ using the transitions in $G$.

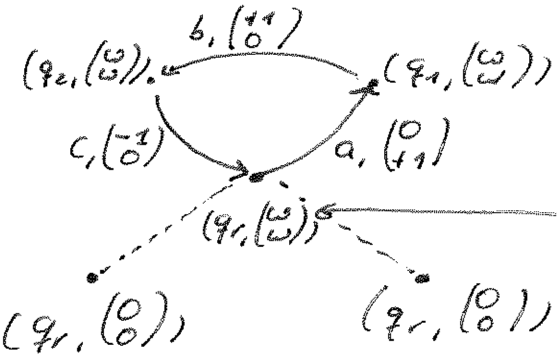
(2) Break it up into MGTS over its basic paths that reach configurations compatible with exit config: same state + specialization preorder.

repeat no configuration

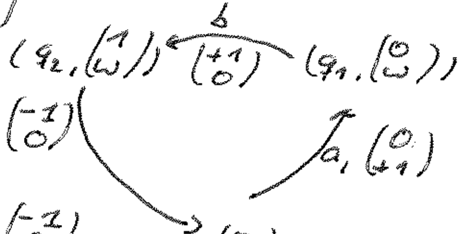
(3) Finally, fix the output marking.

**Remark:** We defined coverability graphs for Petri nets.
The only difference for VASS we the states.
We include them in the ordering:
$(q, c) \leq (p, d)$, if $q = p$ and $c \leq d$.

**Illustration:**



No covering sequence,
the only loop has no effect
on the first counter.



(1)

$$\Rightarrow \begin{tikzcd} (q_r, \binom{0}{0}) \arrow[r, "{a, \binom{0}{+1}}"] & (q_1, \binom{0}{1}) \arrow[r, "{b, \binom{+1}{0}}"] & (q_2, \binom{1}{1}) \arrow[r, "{c, \binom{-1}{0}}"] & (q_r, \binom{0}{\omega}) \arrow[lll, dashed, bend left, "\leq"] \end{tikzcd}$$

(2)

$\Rightarrow$ There are only two paths that
- repeat no configuration and
- end in a configuration $(q, c)$ with $q = q_r$ and $G.Cout \leq_\omega c$.

namely

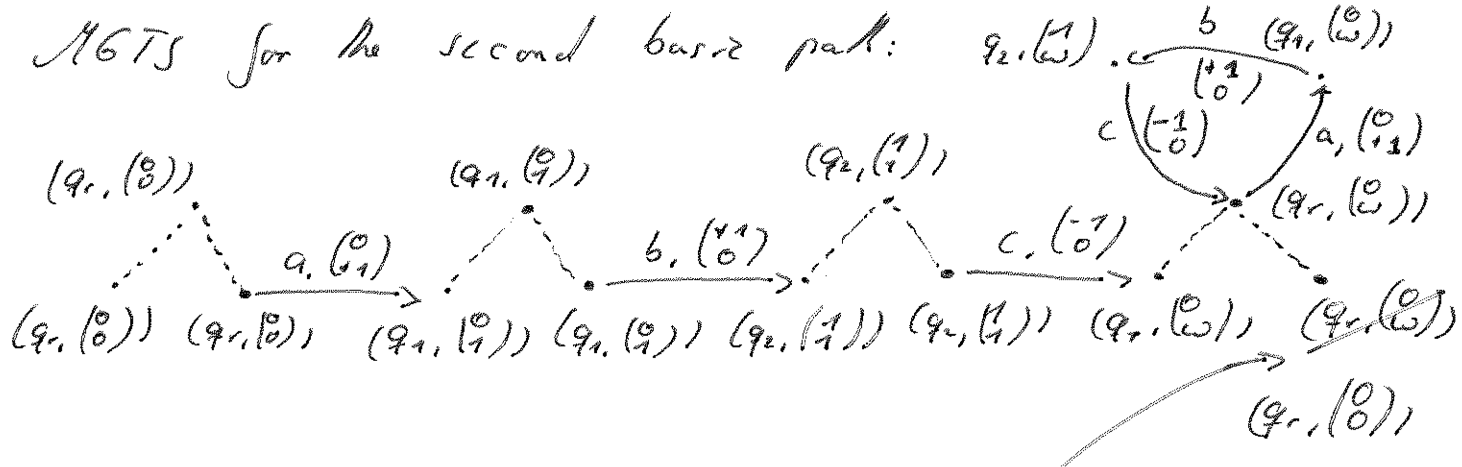
$$\varepsilon, \quad (q_r, \binom{0}{0}) \xrightarrow{a, \binom{0}{+1}} (q_1, \binom{0}{1}) \xrightarrow{b, \binom{+1}{0}} (q_2, \binom{1}{1}) \xrightarrow{c, \binom{-1}{0}} (q_r, \binom{0}{\omega}).$$

We turn each basic path into an MGTS
by replacing each configuration
with the precovering graph of its SCC:

MGTS for $\varepsilon$:

} this is a trivial SCC consisting only of the node

MGTS for the second basic path:



(3)
⟹ Replace the exit marking
of the last precovering graph
(this is c with $G.cout \leq_\omega c$)
by G.cout:

<u>Formally</u>: Let G be a precovering graph.

Let $C = G.C$ and $\Sigma = G.\Sigma$.

The construction of $\mathbb{W}$ is as follows.

1.) Construct the coverability graph

$$H = (W, (q_r, G.cin), E) \text{ of } G$$

with $W \subseteq G.V \times \mathbb{N}_\omega^C$.

2.) Now, we define precovering graphs $G_s$ that work on the SCC in H of $s \in W$.

Let

$$SCC_s := \{ t \in W \mid s \longrightarrow^* t \longrightarrow^* s \text{ in } H \}$$

be the nodes in the SCC of s.

Let

$$V_s := ( SCC_s, \Sigma, C, \underbrace{E|_{SCC_s}}_{\text{Restriction of } E} )$$

With this, the precovering graph is

$$G_S := (V_S, (s,c), (s,c), \mathcal{C}).$$

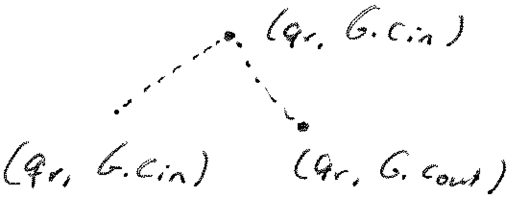
The entry and exit configuration is $(s,c)$ with $s = (q,c)$.

The consistent assignment is taken from $H$:

$$\mathcal{C}((q,x)) = x \in \mathbb{N}_\omega^C \quad \text{f.a.} \quad (q,x) \in SCC_S.$$

3.) For each basic path $\rho$ in $H$
that ends in a configuration $(q,c)$
with $q = G.q_r$ and $G.c_{out} \leq_\omega c$
construct the MGTS $U_\rho$ as follows:

$U_\varepsilon :=$



$$U_{s.up.\rho} := G_S . up . U_\rho .$$

4.) Replace $U.c_{out}$ by $G.c_{out}$.
Add the resulting $U'$ to a set $\mathbb{U}$.

Definition:

$$\mathbb{W} := \{ W[U'/G] \mid U' \in \mathbb{U} \}.$$

Lemma:

For all $U' \in \mathbb{U}$: $[\![Acc_{\mathbb{Z}}(U')]\!] \subseteq [\![Acc_{\mathbb{Z}}(G)]\!]$.

Remark:
Strictly speaking, this is up to a renaming of nodes.

Proof:
We have $Runs_{\mathbb{Z}}(U') \subseteq Runs_{\mathbb{Z}}(G)$.
The reason is that $U'$ only simulates transitions of $G$ from $G.c_{in}$, and hence $G$ can imitate every run in $U'$ exactly.

Since

$$\mathit{IAcc}_{\mathbb{Z}}(U') \subseteq \mathit{Runs}_{\mathbb{Z}}(U') \subseteq \mathit{Runs}_{\mathbb{Z}}(G),$$

we only need to show that

$$\sigma \in \mathit{IAcc}_{\mathbb{Z}}(U')$$

has appropriate initial and final configurations
as required by $G$.
This, however, holds by the construction of $U'$. □

We also need to show
that we do not lose $\mathbb{N}$-runs.

<u>Lemma:</u>

$$\mathit{IAcc}_{\mathbb{N}}(\mathbb{U}) = \mathit{IAcc}_{\mathbb{N}}(G).$$

<u>Proof:</u>

- The inclusion $\mathit{IAcc}_{\mathbb{N}}(\mathbb{U}) \subseteq \mathit{IAcc}_{\mathbb{N}}(G)$
  follows with the above argument.
- For $\mathit{IAcc}_{\mathbb{N}}(G) \subseteq \mathit{IAcc}_{\mathbb{N}}(\mathbb{U})$,
  consider $\sigma \in \mathit{IAcc}_{\mathbb{N}}(G)$.
  The overall argument is that
  the coverability graph $H$
  only discards runs that
  become negative in a counter.
  Such runs, however, are already
  excluded from $\mathit{IAcc}_{\mathbb{N}}(-)$.
- This means there is a basic path $\rho$ in $H$,
  so that $\sigma \in \mathit{Runs}_{\mathbb{N}}(U_\rho)$.

- We moreover have

$$\sigma \in ITAcc_{IN}(U_p).$$

To see this, note that the entry and exit constraints stem from concrete counter values in the coverability graph. As $\sigma$ is a run through the coverability graph, it reaches precisely the given (concrete) values. □

It remains to show that $U_p \in \mathcal{U}$ is smaller than $G$ in some order that we will turn into a well-founded order later on.

(For the initial and final configurations, there is nothing to show as $\sigma \in ITAcc_{IN}(G)$ and $U_p$ has the same initial and final configurations as $G$.)

Lemma:

For every $U' \in \mathcal{U}$, for every $G'$ in $U'$, we have $|\Omega(G')| < |\Omega(G)|$.

Proof:

We have $\Omega(G') \subseteq \Omega(G)$.

- Indeed, no counter that is tracked concretely in $G$ can be pumped by following the transitions of $G$.
- Assume $\Omega(G') = \Omega(G)$.

Then, by the construction of $H$, there is a transition sequence to pump all counters in $\Omega(G) \setminus \Omega(G.cin)$. This contradicts $CSup(G) = \emptyset$. □

## 3.4.2 Case (ii).a : $x[G,in,c] \notin supp(Char(W))$

Recall: · If a counter variable $x[G,in,c]$

is not in the support,

it is bounded in the solution space.

· Let $b \in \mathbb{N}$ be the maximal value.

Definition:

$$\mathbb{W} := \{ W[G.cin(c) := k] \mid 0 \leq k \leq b \}$$

· We replace $G.cin(c) = \omega$

by every concrete value between 0 and b.

· If a run is intermediate accepting,

it will satisfy the characteristic equation

and therefore assume a value between 0 and b

for counter c when at G.cin.

Lemma: $IAcc_{\mathbb{Z}}(W) = IAcc_{\mathbb{Z}}(\mathbb{W})$,

and so in particular $IAcc_{\mathbb{N}}(W) = IAcc_{\mathbb{N}}(\mathbb{W})$.

Towards a well-founded measure,

observe that we reduced

the number of $\omega$-entries in the initial marking:

$$|\Omega(c_{in}^{new})| < |\Omega(c_{in})|.$$

## 3.4.2 Case (ii).b : $x[t] \notin supp(Char(W))$

Recall: · If a transition variable $x[t]$

is not in the support,

it is bounded in the solution space.

- Let $b$ be the maximal value.

Approach:

- We turn the precovering graph $G$ that contains $t$ into a VASS $V(G, t, b)$:
  - ↳ there is a single counter $c_t$,
  - ↳ it is initialized to $b$,
  - ↳ the states and transitions are as in $G$,
  - ↳ every occurrence of $t$ decrements $c_t$, all other transitions leave $c_t$ unchanged.
- We compute the state space of $V(G, t, b)$. Note that it is finite as the only counter $c_t$ only decreases.
  Moreover, note that $t$ never occurs in an SCC, as the counter is decreased and cannot increase again.
- From the state space of $V(G, t, b)$ we construct a set of MGTS $\mathcal{U}$ as we have done with the coverability graph in Case (i).

Definition:

$$\mathbb{W} := \{ \omega[\mathcal{U}/G] \mid \mathcal{U} \in \mathbb{U} \}.$$

Lemma:

$$IAcc_{\mathbb{Z}}(\mathbb{W}) = IAcc_{\mathbb{Z}}(\omega), \text{ and so in particular}$$

$$IAcc_{\mathbb{N}}(\mathbb{W}) = IAcc_{\mathbb{N}}(\omega).$$

Towards a well-founded measure

note that the preceding graphs $G'$ in $U' \in \mathcal{U}$ do not contain $E$ (because $E$ does not occur in SCCs, as observed earlier). The remaining transitions in $G'$ are transitions of $G$.

Hence,

$$|G'.T| < |G.T|.$$

## 3.4.3 The Well-Founded Measure

*Recall:*

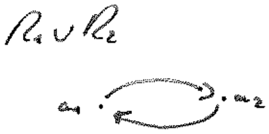
- A relation $R \subseteq A \times A$ is *well-founded*, if there is no infinite sequence

  $$a_0 \; R \; a_1 \; R \; a_2 \; \dots$$

- Well-founded relations are *not* closed under unions:

  $$A = \{a_1, a_2\}, \quad R_1 = \{(a_1, a_2)\}, \quad R_2 = \{(a_2, a_1)\}.$$

  Then $R_1$ and $R_2$ are well-founded, but $R_1 \cup R_2$ is not.



- Well-founded relations are closed under forming *lexicographic orderings*.

  $(A, R_1)$, $(B, R_2)$ well-founded

  $\Rightarrow (A \times B, R_{lex})$ well-founded, where

  $(a_1, b_1) \; R_{lex} \; (a_2, b_2)$, if $a_1 \; R_1 \; a_2$ or $a_1 = a_2$ and $b_1 \; R_2 \; b_2$.

Definition:

- We associate with a precovering graph $G$ its Lambert rank:

$$\text{lrank}(G) := (|\Omega(G)|, |G.T|, |\Omega(G.\text{cin})| + |\Omega(G.\text{cout})|). \in \mathbb{N}^3$$

We compare Lambert ranks lexicographically.

- We define the well-founded measure that decreases with every decomposition step by induction on the structure of MGTS:
  - $G_1 < G_2$, if $\text{lrank}(G_1) <_{lex} \text{lrank}(G_2)$.
  - $W < G$, if $G' < G$ for all $G' \in W$.
  - $W_1 < W_2$, if $W_2 = G_1 \ldots G_k$, $W_1 = W_1' \ldots W_k'$ so that $W_i' < G_i$ for all $1 \leq i \leq k$.

Note:

- The decomposition steps for missing pumping sequences (Case (i)) and for bounded transitions (Case (iii))

may introduce new precovering graphs whose initial and final markings may have more $\omega$-entries than the ones of $G$.
This is why we have to place $|\Omega(G.\text{cin})| + |\Omega(G.\text{cout})|$ last in the lexicographic ordering.

- Similarly, the coverability graph may contain several copies of a transition, even in the same SCC (the coverability graph will track more counters than $G$ concretely).
This is why $|\Omega(G)|$ is placed before $|G.T|$.