6.1 Case (1): $j \notin supp(Chars_{sd}(W))$

<u>Setting</u>:
- $W = (U, \mu)$ faithful but imperfect
- $sol(Chars_{sj}(W)) \neq \emptyset \neq sol(Chars_{dy}(W))$
- G precovering graph in W, $j \in sd$, $c_{io} \in \{G.c_{in}, G.c_{out}\}$ with $c_{io}[j] = \omega$
- $x[G, c_{io}, j] \notin supp(Chars_{sd}(W))$

<u>Insight</u>: If the variable is not in the support, the set

$$A_{sd} := \{ s[G, c_{io}, j] \mid s \in sol(Chars_{sd}(W))\}$$

is
- ↳ finite,
- ↳ non-empty, due to $sol(Chars_{sd}(W) \neq \emptyset)$,
- ↳ $\subseteq \mathbb{N}$ by the shape of $Chars_{sd}(W)$.

<u>Goal</u>: Construct $(X, Y) = dec(W)$ with
(a) $\forall W' \in X.$ W' faithful and $W' < W$
(b) $\forall W' \in Y.$ $L_{sj}(W') \mid D_n$
(c) $L_{sj}(W) = L_{sj}(X \cup Y)$.

6.1.1 Case $sd = sj$

Define

$$X := \{ (U_a, \mu) \mid a \in A_{sj} \} \qquad Y := \emptyset.$$

By U_a, we mean U with the value of j in c_{io} changed from ω to a.

Properties:

(c) $L_{sj}(X \cup Y) \subseteq L_{sj}(W)$,

because concrete values

make intermediate acceptance stricter.

$L_{sj}(W) \subseteq L_{sj}(X \cup Y)$

A run $\rho \in IAcc_{sj}(W)$ solves $Chw_{sj}(W)$.

Hence, it reaches c_{io} with a value in j

that belongs to A_{sj}.

(b) As $Y = \emptyset$, there is nothing to show.

(a) Faithfulness:

We neither changed the edges nor the dy-markings.

Therefore, faithfulness follows from

the faithfulness of W.

Decrease:

$\Omega(G)$, $G.E$, $G.C_{\overline{IO}}$ stay unchanged

and we reduce $|\Omega(c_{io})|$.

6.1.2 Case sd = dy

Note: • This is the complicated case

where we change an entry or exit marking

for the Dyck counters from ω to $a \in \mathbb{N}$.

• As a consequence, we have to check faithfulness.

Approach: The plan is to use the following trick.

<u>Lemma 1</u>: Consider $0 \leq a, b < \mu_{new}$.

If $b \equiv a \mod \mu_{new}$, then $b = a$.

(i) In our application, $b \in Ady$.

Hence, we have to make sure

$\boxed{b < \mu_{new}}$.

(ii) We cannot simply increase μ_{new}

to be larger than all values in Ady.

The problem is that we need

<u>acceptance modulo μ_{new}</u>

$\implies$ <u>acceptance modulo μ</u>

This works if μ <u>divides</u> μ_{new}.

We thus pick

$\boxed{\mu_{new} := \mu \cdot l}$

We explain in a moment

the value of l that has to be chosen.

(iii) If we change μ to μ_{new}

we have to adapt the entry and exit valuations

of all precovering graphs.

To see this, consider

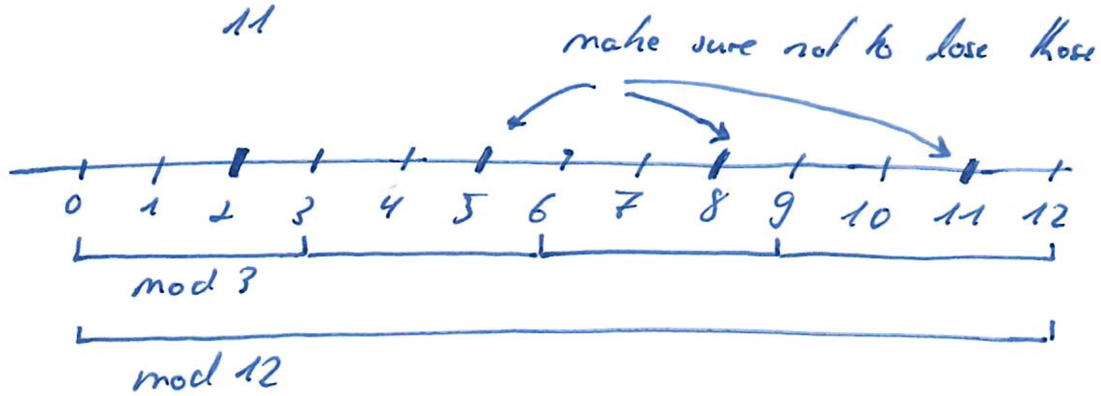
$x \equiv 2 \mod \underset{\mu}{3}$

Let $l = 4$ and thus $\mu_{new} = 12$

Then $x \equiv 2 \mod 12$

will <u>not</u> capture all values of x
that satisfy the first congruence.
Instead, we need the values for all

$$x \equiv \begin{matrix} 2 \\ 5 \\ 8 \\ 11 \end{matrix} \mod 12$$



<u>Lemma 2</u>:

Let μ divide μ_{new}.
Consider $x, h \in \mathbb{Z}$.
Then

$$x \equiv h \mod \mu$$

$$\text{iff } \exists\, 0 \leq i < \mu_{new}.\ x \equiv i \mod \mu_{new} \ \wedge\ i \equiv h \mod \mu.$$

To adapt the entry and exit markings,
we define an equivalence $S_1 \equiv_\mu S_2$ on MGTS
that allows us to change h to i
with $i \equiv h \mod \mu$
as in Lemma 2.

<u>Definition</u>:

We define $S_1 \equiv_\mu S_2$ by induction
on the structure of MGTS:

↳ $G_1 \equiv_\mu G_2$, if
- $G_1.V = G_2.V$
- $G_1.E = G_2.E$
- $G_1.v_{root} = G_2.v_{root}$
- $G_1.c_{IO}[sj] = G_2.c_{IO}[sj]$
- $G_1.c_{IO}[dy] \equiv G_2.c_{IO}[dy] \mod \mu$.

↳ $S_1.up.S_2 \equiv_\mu S_1'.up.S_2'$, if
- $S_1 \equiv_\mu S_1'$
- $S_2 \equiv_\mu S_2'$.

We write $0 \leq S < c$ to mean all entry and exit markings in all precovering graphs of S take values from $[0, c-1] \cup \{\omega\}$.

- The following lemma lifts Lemma 2 to intermediate accepting computations.

Lemma 3:

Let μ divide μ_{new}.

Then

$$IAcc_{SJ}(\mathcal{U}, \mu) = \bigcup_{\substack{V \equiv_\mu \mathcal{U} \\ 0 \leq V < \mu_{new}}} IAcc_{SJ}(V, \mu_{new}).$$

Note:
- $V \equiv_\mu \mathcal{U}$ corresponds to $c \equiv h \mod \mu$ in Lemma 2.
- $0 \leq V < \mu_{new}$ corresponds to $0 \leq i \leq \mu_{new}$ in Lemma 2.
- The union corresponds to the existential quantifier.

(iv) If we change the entry and exit values
in all precovering graphs,
we have to recheck faithfulness.
The idea is to use the trick in Lemma 1.
To be able to do so,

μ_{new} has to be larger
than all entry and exit values
in all precovering graphs.

Note that we do not maintain the invariant
that μ is larger than all these values.
This would complicate the decomposition
in the remaining cases:

if we unwind a precovering graph into an MGTS
and introduce new intermediate values,
we would have to repeat the argumentation here
also there.

We thus select

$l :=$ least value larger than
- all values in Δ_{dy}
- all concrete values in entry and exit markings of all precovering graphs.

Definition:

$Z := \{(S, \mu_{new}) \mid S \cong_\mu U_a,\ 0 \leq a < \mu_{new},\ 0 \leq S < \mu_{new},$
$S.cin[dy] = 0\}$

$X := \{(S, \mu_{new}) \in Z \mid S.cout[dy] = 0\}$

zero-reaching

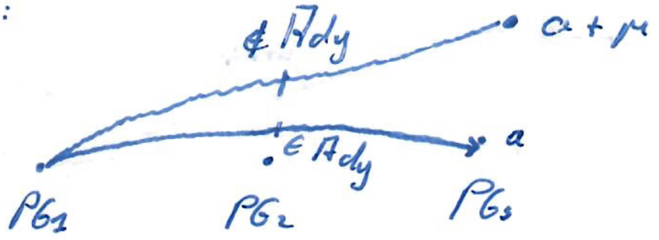
$Y := Z \setminus X.$

Remark:

- Note that we not only change the value of counter j in C_{io} from ω to a value in Ady.
- Instead, we consider all values from 0 to $\mu_{new} - 1$.
- The reason is that Ady is constructed from $Chw_{dy}(W)$. These equations capture ($\mathbb{Z}$-) runs that read intermediate Dyck values precisely.
- The language $L_{sj}(W)$ that we wish to preserve, however, only checks intermediate Dyck values modulo μ.
- This means Ady may not contain enough values to guarantee $L_{sj}(W) = L_{sj}(X \cup Y)$.

Remark: We need $0 \leq a < \mu_{new}$ for finiteness of X and Y.

Illustration:



Proof of Properties (a) to (c):

(c) $L_{sj}(X \cup Y) \subseteq L_{sj}(W)$.

Consider $W_{new} = (S, \mu_{new}) \in X \cup Y$.

Then

$$IAcc_{sj}\underbrace{(S, \mu_{new})}_{W_{new}} \overset{\text{Lemma 3}}{\subseteq} IAcc_{sj}(U_a, \mu) \subseteq IAcc_{sj}\overbrace{(U, \mu)}^{W}$$

$0 \leq S < \mu_{new}$, $S \equiv_\mu U_a$

Concrete values make intermediate acceptance stricter

$L_{sj}(W) \subseteq L_{sj}(X \cup Y)$.

Consider $\rho \in IAcc_{sj}(W)$.

- As we do not change the entry and exit markings for the counters in sj,

we have

$$\rho \in IAcc_{sj, \sqsubseteq_\omega[sj]}(W_{new}) \text{ for all } W_{new} \in X \cup Y.$$

- To see $\rho \in IAcc_{dy, \sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new})$ for some $W_{new} \in X \cup Y$,

assume ρ reaches c_{i0} with value b in counter j.

Let

$$b \equiv a \mod \mu.$$

We now have $\rho \in IAcc_{dy, \sqsubseteq_\omega^{\mu}[dy]}(U_{a,\mu})$.

- To get from (U_a, μ) to some $W_{new} \in X \cup Y$,

we have to change μ to μ_{new} and hence the intermediate values of all precovering graphs.

By Lemma 3, there is $S \equiv_\mu U_a$, $0 \leq S < \mu_{new}$,

with $\rho \in IAcc_{dy, \sqsubseteq_\omega^{\mu_{new}}[dy]}(S, \mu_{new})$.

Note that $(S, \mu_{new}) \in X \cup Y$.

(b) Consider $W_{new} = (S, \mu_{new}) \in Y$

for which we want to show $L_{sj}(W_{new}) \mid D_n$.

By definition of Y,

$$S.c_{in}[dy] = 0 \quad \text{and} \quad 0 < S.c_{out}[dy] < \mu_{new}.$$

The initial and final values are all concrete,

because W is zero-reaching by faithfulness.

- With these initial and final values,
  every run
  $$\rho \in IAcc_{sj}(W_{new}) \subseteq IAcc_{\mathbb{Z},\, \sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new})$$
  has an effect
  $$c \not\equiv 0 \mod \mu_{new}$$
  on the Dyck counters.
- Then, $\lambda(\rho)$ will have the same effect on D_n
  by Dyck-visibility.
- Hence, an NFA that tracks
  the Dyck counters modulo μ_{new}
  shows
  $$L_{sj}(W_{new}) \mid D_n.$$

(a) Decrease:

The argument is the same as for $sd = sj$.

Faithfulness:

Consider $W_{new} = (S, \mu_{new}) \in X$.

Then W_{new} is zero-reaching by the definition of X.

For
$$\boxed{\begin{aligned} &IAcc_{\mathbb{Z},dy}(W_{new}) \cap IAcc_{\mathbb{Z},\, \sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new}) \\ &\qquad \subseteq IAcc_{\mathbb{Z},dy}(W_{new}) \end{aligned}},$$

we use

(1) $$IAcc_{\mathbb{Z},dy}(W_{new}) \cap IAcc_{\mathbb{Z},\, \sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new}) \subseteq IAcc_{\mathbb{Z},dy}(\underline{\underline{W}})$$

(2) $$IAcc_{\mathbb{Z},dy}(W) \cap IAcc_{\mathbb{Z},\, \sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new}) \subseteq IAcc_{\mathbb{Z},dy}(W_{new}).$$

It remains to show Inclusions (1) and (2).

Inclusion (1):

We use

- $\mathrm{Acc}_{\mathbb{Z},dy}(W_{new}) \subseteq \mathrm{Acc}_{\mathbb{Z},dy}(W)$
- $\mathrm{IAcc}_{\mathbb{Z},\sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new}) \subseteq \mathrm{IAcc}_{\mathbb{Z},\sqsubseteq_\omega^{\mu}[dy]}(W)$
- and the faithfulness of W.

For

$$\mathrm{Acc}_{\mathbb{Z},dy}(W_{new}) \subseteq \mathrm{Acc}_{\mathbb{Z},dy}(W)$$

note that we only changed an intermediate value which acceptance does not see

For

$$\mathrm{IAcc}_{\mathbb{Z},\sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new}) \subseteq \mathrm{IAcc}_{\mathbb{Z},\sqsubseteq_\omega^{\mu}[dy]}(W)$$

we use the argument from (c) above.

Inclusion (2):

Consider $\rho \in \mathrm{IAcc}_{\mathbb{Z},dy}(W) \cap \mathrm{IAcc}_{\mathbb{Z},\sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new})$.

Consider the moment the run reaches c_{i0}

and consider counter j whose value

we changed from ω to $a \in [0, \mu_{new} - 1]$.

- By $\rho \in \mathrm{IAcc}_{\mathbb{Z},dy}(W)$,

the run solves $\mathrm{Chw}_{dy}(W)$

and hence reaches a value $b \in \mathrm{Ady}$

at the moment of interest.

Note that $0 \leq b < \mu_{new}$, as $\ell > \max \mathrm{Ady}$.

- By $\rho \in \mathrm{IAcc}_{\mathbb{Z}, \sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new})$,

  we have

  $$b \equiv a \mod \mu_{new}.$$

- Since $0 \leq a, b < \mu_{new}$,

  we conclude $b = a$ by Lemma 1.

- For the remaining precovering graphs,
  and for precovering graph G but a counter different from j,
  consider

$$\underbrace{\rho \in \underset{\substack{\text{reaches } 0 \leq b < \mu_{new} \\ \text{by the choice of } l}}{\mathrm{IAcc}_{\mathbb{Z}, dy}(W)} \cap \underset{\substack{\text{reaches } b' \text{ modulo } \mu_{new} \\ \text{with } 0 \leq b' < \mu_{new} \\ \text{on } 0 \leq S < \mu_{new}.}}{\mathrm{IAcc}_{\mathbb{Z}, \sqsubseteq_\omega^{\mu_{new}}[dy]}(W_{new})}}_{\text{at the moment of interest.}}$$

We thus have

$$b \equiv b' \mod \mu_{new} \quad \text{and} \quad 0 \leq b, b' < \mu_{new}.$$

and so

$$b = b' \text{ by Lemma 1.}$$

$\square$