# 5. Separability Transfer

## 5.1 Regular Separability

Goal: Show that $L_{\mathbb{Z},sj}(W) \mid L_{\mathbb{Z},dy}(W) \Rightarrow L_{sj}(W) \mid D_n$,

provided $W$ is faithful.

Approach: Modify the given separator.

Let $B^{\#}$ separate $L_{\mathbb{Z},sj}(W)$ and $L_{\mathbb{Z},dy}(W)$.

Let $B$ be $B^{\#}$ without $\#$ symbols ($(a,\#) \rightsquigarrow a$).

We want to use $B$ as separator for $L_{sj}(W)$ and $D_n$.

Definition:

$B^{\#}$ is precise, if

$$L(B^{\#}) \cap D_n^{\#} \subseteq L_{\mathbb{Z},dy}(W).$$

($D_n^{\#}$: $D_n$ with $a$ and $(a,\#)$ everywhere)

Idea:

- $B^{\#}$ only intersects $D_n^{\#}$ along $W$.
- This cannot happen as $B^{\#}$ is a separator.

Lemma:

If $B^{\#}$ separates $L_{\mathbb{Z},sj}(W)$ and $L_{\mathbb{Z},dy}(W)$ and is precise, then $B$ separates $L_{sj}(W)$ and $D_n$.

Insight: Every separator can be made precise.

Lemma: Let $W$ be faithful.

Every separator $B^{\#}$ of $L_{\mathbb{Z},sj}(W)$ and $L_{\mathbb{Z},dy}(W)$ can be turned into a precise separator $B^{\#} \times A^{\#}$.

The NFA $A^{\#}$ is independent of $B^{\#}$.

## Discussion:

First failure of preciseness:

- $B^\#$ does not follow the control flow of $W$.
- Understand $W$ as NFA $B^\#(W)$, use this as $A^\#$.
- Assume $D_n^\# \cap L(B^\#)$ is $L(B^\#(W)) \neq \emptyset$.

  Then the word labels
  - ↳ a run $\rho$ through $W$
  - ↳ that takes the Dyck counter from $\bar{0}$ to $\bar{0}$ (visibility).

  $\Rightarrow \rho \in Acc_{dy}(W)$.

Second failure of preciseness:

- $L_{\mathbb{Z}, dy}(W)$ is not defined via $Acc_{\mathbb{Z}, dy}(W)$ but via $IAcc_{\mathbb{Z}, dy}(W)$.
- $B^\#(W)$ does not give a guarantee that we read intermediate counter values.

Faithfulness:

$$Acc_{\mathbb{Z}, dy}(W) \cap IAcc_{\mathbb{Z}, \Xi^n_\omega[dy]}(W) \subseteq IAcc_{\mathbb{Z}, dy}(W).$$

- We need $A^\#$ to
  - ↳ follow the control flow of $W$
  - ↳ maintain the Dyck counter modulo $\mu$
  - ↳ check intermediate acceptance modulo $\mu$ at intermediate markings.

Note:

- For $B^\# \times A^\#$ to be a separator, we still have to cover $L_{sj}(W)$.

  This works

  thanks to $\boxed{\cap\ \mathcal{I}Acc_{dy,\, \sqsubseteq^{\mu}_{\omega}[dy]}(W)}$

  in the definition of $\mathcal{I}Acc_{sj}(W)$.

  ⇝ The intersection allows us to restrict a given separator without losing words.

- $A^\#$ alone may not be separator.
- $\#$ is not needed for this part of the separability transfer lemma.

Definition:

(sj ⊎ dy), we could just have done with dy.

$$A^\# := (Q \times [0, \mu-1]^d,\ \Sigma_n \times \{\varepsilon, \#\},\ \delta,\ ((G_{first}, in), (c_1, \bar{0}) \bmod \mu),\ ((G_{last}, out), (c_2, \bar{0}) \bmod \mu)).$$

↳ $Q$ = all states in $W$, $(G, in), (G, out)$ for every precovering graph $G$ in $W$.

↳ $\delta$ =

- (within precovering graphs) $(v, x) \xrightarrow{a} (w, x+y \bmod \mu)$ for all $(v, a, y, w)$ in a precovering graph of $W$, and all $x \in [0, \mu-1]^d$.
- (entering and leaving precovering graphs) $((G, in), x) \xrightarrow{\varepsilon} (G.v_{root}, x)$, if $x \sqsubseteq^{\mu}_{\omega} G.c_{in}$
- $((G, out), x) \xrightarrow{\varepsilon} (G.v_{root}, x)$, if $x \sqsubseteq^{\mu}_{\omega} G.c_{out}$
- (between neighboring precovering graphs) $((G, out), x) \xrightarrow{\lambda\#(\rho)} ((G', in), x + \mathit{eff}(\rho) \bmod \mu)$.