4.3 Deciding Regular Separability

Approach: Decompose an initial DMGTS W until the regular separability

$$L_{sj}(W) \mid D_n$$

boils down to

$$L_{\mathbb{Z},sj}(W') \mid L_{\mathbb{Z},dy}(W') \quad \text{for all } W' \text{ in the decomposition.}$$

Behind this are two lemmas.

Lemma 1 (Decomposition):

We can decompose a faithful DMGTS W into two finite sets

Perf and Fin of DMGTS,

where
- all $S \in$ Perf are perfect
- all $T \in$ Fin satisfy $L_{sj}(T) \mid D_n$
- $L_{sj}(W) = L_{sj}(\text{Perf} \cup \text{Fin})$.

Lemma 2 (Separability Transfer):
- If W is faithful, then
$$L_{\mathbb{Z},sj}(W) \mid L_{\mathbb{Z},dy}(W) \Rightarrow L_{sj}(W) \mid D_n.$$
- If W is perfect, then
$$L_{\mathbb{Z},sj}(W) \not\mid L_{\mathbb{Z},dy}(W) \Rightarrow L_{sj}(W) \not\mid D_n.$$

Recall that perfect DMGTS are faithful.

<u>Corollary</u>: Let $\mathcal{W}$ be perfect.

Then

$$L_{\mathbb{Z},sj}(\mathcal{W}) \mid L_{\mathbb{Z},dy}(\mathcal{W}) \quad \text{iff.} \quad L_{sj}(\mathcal{W}) \mid D_n.$$

The two lemmas give us the result we are after.

<u>Proposition</u>:

Let $\mathcal{U}$ be an initialized VRSS over Σ_n.

Then $L(\mathcal{U}) \mid D_n$ is decidable.

<u>Proof</u>:

- Let $\mathcal{U} = ((\underbrace{\{v\}, \Sigma_n, sj, E}_{\text{VRTS}}), \overbrace{(v, c_1), (v, c_2)}^{\text{Counters}})$

  We assume to have a VRTS.

  Every VRSS can be turned into a VRTS with the same language.

- Check $L(\mathcal{U}) \cap D_n = \emptyset$.

  If not, return false.

- Otherwise, we construct an initial DMGTS $\mathcal{W}$ that is faithful and satisfies $L_{sj}(\mathcal{W}) = L(\mathcal{U})$.

We can now check $L_{sj}(\mathcal{W}) \mid D_n$.

- Since $\mathcal{W}$ is faithful, we can invoke Lemma 1.

  It yields finite sets Perf and Fin of DMGTS with $L_{sj}(\mathcal{W}) = L_{sj}(\text{Perf} \cup \text{Fin})$.

Hence, to decide

$$L_{sj}(W) \mid D_n$$

we can check

$$L_{sj}(S) \mid D_n \quad \text{for all } S \in Perf \text{ and}$$
$$L_{sj}(J) \mid D_n \quad \text{for all } J \in Fin.$$

- For $J \in Fin$, we already have

$$L_{sj}(J) \mid D_n \quad \text{by Lemma 1.}$$

- For $S \in Perf$, we have

$$L_{sj}(S) \mid D_n \quad \text{iff} \quad L_{\mathbb{Z},sj}(S) \mid L_{\mathbb{Z},dy}(S).$$

Our algorithm checks the latter.
Using the lemma from last lecture,
we compute $\mathbb{Z}$-VASS V_1 and V_2

with
$$L_{\mathbb{Z}}(V_1) = L_{\mathbb{Z},sj}(S)$$
$$L_{\mathbb{Z}}(V_2) = L_{\mathbb{Z},dy}(S).$$

Then

$$L_{sj}(S) \mid D_n \quad \text{iff} \quad L_{\mathbb{Z}}(V_1) \mid L_{\mathbb{Z}}(V_2).$$

With [ICALP '17], we check $L_{\mathbb{Z}}(V_1) \mid L_{\mathbb{Z}}(V_2)$.

- If the check fails for some $S \in Perf$,
we return false.
Otherwise, we return true. $\square$

Construction of the initial DMGTS $\mathcal{W}$:

We define $\mathcal{W} = (G, \mu)$ with $\mu = 1$.

The precovering graph G has the underlying VASS

$$V = (\{v_{root}\}, \Sigma_{n}, sj \uplus dy, E').$$

The edges are

$(v_{root}, a, (x, y), v_{root})$ for every $(v, a, x, v) \in E$.

Here, y modifies the Dyck counters as required by Dyck visibility.

We set

$$G = (V, (v_{root}, (c_1, \bar{0})), (v_{root}, (c_2, \bar{0})), \mathcal{C}).$$

All counters are decorated ω, $\mathcal{C} = \mathbb{N}^{\emptyset}$.

Claim 1: $L_{sj}(\mathcal{W}) = L(\mathcal{U})$

The only difference between $\mathcal{W}$ and $\mathcal{U}$ is that $\mathcal{W}$ requires acceptance modulo $\mu = 1$ for the Dyck counters.

This, however, is no restriction as $h \equiv 0 \mod 1$ for all $h \in \mathbb{N}$.

Claim 2: $\mathcal{W}$ is faithful

- The initial and final marking are zero on the Dyck counters, so $\mathcal{W}$ is zero reading.
- For $Acc_{\mathbb{Z},dy}(\mathcal{W}) \cap IAcc_{\mathbb{Z}, E^{\mu}_{\omega}[dy]}(\mathcal{W}) \subseteq IAcc_{\mathbb{Z},dy}(\mathcal{W})$, we note that
$$Acc_{\mathbb{Z},dy}(\mathcal{W}) = IAcc_{\mathbb{Z},dy}(\mathcal{W})$$
since there are no intermediate precovering graphs/intermediate markings.