4.2 DMGTS

Goal: Introduce doubly-marked graph transition sequences (DMGTS) as the data structure underlying our decision procedure.

Idea:
- Simultaneously track both languages, the language of the subject VASS and the Dyck language.
- Similar to intersection but the coupling is not as tight. Instead, a DMGTS defines two languages, one for the subject VASS and one for the Dyck VASS.
- The decision procedure then decomposes an initial DMGTS until a notion of perfectness holds, then separates $\mathbb{Z}$-versions of the two languages, and lifts the separator to the languages of interest.
- The decomposition and also perfectness do not treat the languages as symmetric. The focus is on the subject language that a separator has to cover.
- To achieve this, the subject language has to maintain approximate information about the Dyck language — new Technique 1
- This is closely related to the notion of faithfulness — new Technique 2.

Definition:
- A DMGTS is a pair $W = (S, \mu)$ consisting of an MGTS S and a natural number $\mu \geq 1$.

The counters in S form a disjoint union $sj \uplus dy$

counters in the subject VASS — Dyck counters.

The alphabet is Σ_n and the counters in dy are updated in a <u>visible</u> way.

We use sd to stand for sj or dy.

- The <u>well-founded relation</u> is inherited from MGTS,

$$(S_1, \mu_1) < (S_2, \mu_2), \text{ if } S_1 < S_2.$$

- The <u>size</u> is $|W| := |S| + |\mu|$ binary

- The DMGTS is <u>zero-reaching</u>, if $W.\mathrm{cin}[dy] = 0 = W.\mathrm{cout}[dy]$.

<u>Idea</u>: · Different from intersection we define <u>two languages</u> on a DMGTS.

The subject language is defined via runs that may not be accepting for the Dyck counters.

The Dyck language is defined via runs that do not even consider the subject counters (they may even fall below zero).

<u>New Technique</u>: · The subject language of the DMGTS also has to consider the Dyck counters, but only in an approximate way.

· It checks <u>acceptance</u> of dy <u>modulo μ</u>, where μ is the number from the definition of DMGTS.

· <u>Faithfulness</u>, a new notion for DMGTS, will then guarantee that if the dy-side is accepting modulo μ, then it is accepting in the normal sense.

<u>Definition</u>:

- The <u>modulo-μ specialization order</u>

$$\sqsubseteq_\omega^\mu \subseteq \mathbb{Z}_\omega \times \mathbb{Z}_\omega$$

is defined by

$$i \sqsubseteq_\omega^\mu j, \quad \text{if} \quad j = \omega \quad \text{or} \quad i \equiv j \mod \mu.$$

- $\sqsubseteq[sd]$ = restriction of preorder $\sqsubseteq$ to the counters in sd.

- <u>sd-counter valuation</u>: $\mathbb{N}_\omega^{sd} \times \mathbb{Z}_\omega^{sd}$

<u>sd-run</u>: only sd-counter valuations.

- $Acc_{sd,\sqsubseteq}$ = all sd-runs ρ where

$$0 \leq \rho[first] \sqsubseteq \text{initial configuration}$$
$$0 \leq \rho[last] \sqsubseteq \text{final configuration}.$$

$IAcc_{sd,\sqsubseteq}$ = similar, but for intermediate acceptance.

<u>Definition</u>:

Let W be a DMGTS.

$$(I)Acc_{dy}(W) := (I)Acc_{dy, \underbrace{\sqsubseteq_\omega[dy]}_{\substack{\text{specialization}\\ \text{order on dy}}}}(W)$$

$$(I)Acc_{\mathbb{Z},dy}(W) := (I)Acc_{\mathbb{Z}, \sqsubseteq_\omega[dy]}(W)$$

as usual

$$IAcc_{sj}(W) := IAcc_{sj, \sqsubseteq_\omega[sj]}(W) \boxed{\cap\, IAcc_{dy, \sqsubseteq_\omega^\mu[dy]}(W)}$$

$$IAcc_{\mathbb{Z},sj}(W) := IAcc_{\mathbb{Z}, \sqsubseteq_\omega[sj]}(W) \boxed{\cap\, IAcc_{\mathbb{Z}, \sqsubseteq_\omega^\mu[dy]}(W)}$$

new technique.

With this,

$$L_{sd} := \{ \lambda(\rho) \mid \rho \in IAcc_{sd}(W) \}$$

$$L_{\mathbb{Z},sd} := \{ \lambda_\#(\rho) \mid \rho \in IAcc_{\mathbb{Z},sd}(W) \}.$$

The function $\lambda_{\#}$ extracts the letters on transitions,
except for updates between precovering graphs.
There, we use

$$\lambda_{\#}(up) := (\lambda(up), \#).$$

The $\#$ symbols will allow us to detect
the current precovering graph.

Remark:

Assume we define

$$IAcc'_{sj}(W) := IAcc_{sj, \sqsubseteq_\omega[sj]}(W) \cap IAcc_{\mathbb{Z}, \sqsubseteq^{\mu}_{dy}[dy]}(W),$$

meaning the Dyck counters may become negative.
Then still

$$L_{sj}(W) = L(IAcc'_{sj}(W)),$$

so the language does not change.
The reason is that modulo-μ acceptance
is <u>periodic in μ</u>.
If we have an accepting run ρ,
we can scale the valuation of some counters
consistently in all configurations of ρ by μ
and still arrive at an accepting run.
We use $\rho + \mu$ for the run
in which we scale all counters.

Plan: Invoke the $\mathbb{Z}$-REGSEP decision procedure from [ICALP '17]
on $L_{\mathbb{Z}, sj}(W)$ and $L_{\mathbb{Z}, dy}(W)$.

Needed: Link between $L_{\mathbb{Z}, sd}(W)$ and $\mathbb{Z}$-VASS.

Lemma:

Assume W is zero-reaching.

- Then $L_{dy}(W) \subseteq D_n$.
- Moreover, one can construct a $\mathbb{Z}$-VASS V with $L_{\mathbb{Z}}(V) = L_{\mathbb{Z},sd}(W)$.

We also need characteristic equations.

Definition:

$$Chw_{dy}(G,\mu) := RunsEq(G) \left.\right\} \begin{array}{l} Kirch(G) \\ \wedge\ Mark(G) \\ \wedge\ x[G.E] \geq 0 \end{array}$$

$$\wedge\ IAccEq(G, E_\omega[dy]) \left.\right\} \begin{array}{l} 0 \leq x[G, in] \sqsubseteq_{E_\omega} G.cin \\ \wedge\ 0 \leq x[G, out] \sqsubseteq_{E_\omega} G.cout. \end{array}$$

$$Chw_{sj}(G,\mu) := RunsEq(G)$$
$$\wedge\ IAccEq(G, E_\omega[sj])$$
$$\wedge\ IAccEq(G, E_\omega^\mu[dy]).$$

$$Chw_{sd}(G.up.W,\mu) := Chw_{sd}(G,\mu) \wedge Chw_{sd}(W,\mu)$$
$$\wedge\ x[W[first], in] - x[G, out] = eff(up).$$

We again have the support $supp(Chw_{sd}(W,\mu))$ by considering the homogeneous variant of the equations.

We are now prepared to define the new notion of faithfulness.

Idea: If we have $\bar{0}$ on dy in the beginning and in the end, as expected by D_n, and if we accept on the dy counters modulo-μ at intermediate markings

$\Rightarrow$ then we are already intermediate accepting (without modulo).

Definition:

A DMGTS W is *faithful*, if

- it is zero-reaching // dy counters initially and finally zero

and

- $IAcc_{\mathbb{Z}, dy}(W) \cap IAcc_{\mathbb{Z}, \mathbb{E}^{m}_{\omega}[dy]}(W) \subseteq IAcc_{\mathbb{Z}, dy}(W)$

(the reverse trivially holds).

Definition:

A DMGTS W is *perfect*, if

- it is faithful

and

- for every precovering graph G in W

and for every $sd \in \{sj, dy\}$:

↳ $CSup(G) \neq \emptyset \neq CSdown(G)$ // all counters, $sj + dy$

↳ $supp(Chw_{sd}(W))$ justifies

the <u>unboundedness</u> of <u>sd in G</u>

only sd-variables

+ all edges

Remark:

We only need faithfulness for $\mathbb{N}$-runs,

but the above notion is a stronger result

(the $\mathbb{N}$-version follows if we consider

$Acc_{dy}(W)$ or $IAcc_{dy, \mathbb{E}^{m}_{\omega}[dy]}(W)$ to the left).