Lemma:

$f(0) = 1$

nowhere positive } Satisfied by initial vector.
nowhere covering }

Lemma:

$f(i+1) \leq (2^\wedge f(i))^{i+1} + f(i)$ f.a. $0 \leq i < k$.

Proof:

Let $V \in \mathbb{Z}^k$ and $0 \leq i < k$
so that there is an $(i+1)$-bounded, $(i+1)$-covering path
starting in V.

Case 1: There is an $(i+1)$-$(2^\wedge f(i))$-bounded path that is $(i+1)$-covering.

↳ Then there is such a path
that does not repeat vectors.
To be more precise: where the vectors do not repeat
in the first $(i+1)$ places.

↳ Such a non-repeating path has a length
of at most

$(2^\wedge f(i))^{i+1}$.

Why? There are $2^\wedge f(i)$ possible values per entry/dimension
and $(i+1)$ entries.

Case 2: Otherwise

↳ Consider an $(i+1)$-bounded, $(i+1)$-covering path
that is not
$(i+1)$-$(2^\wedge f(i))$-bounded.

↳ The path can be decomposed into

$$\sigma = \sigma_1 . \sigma_2$$

so that

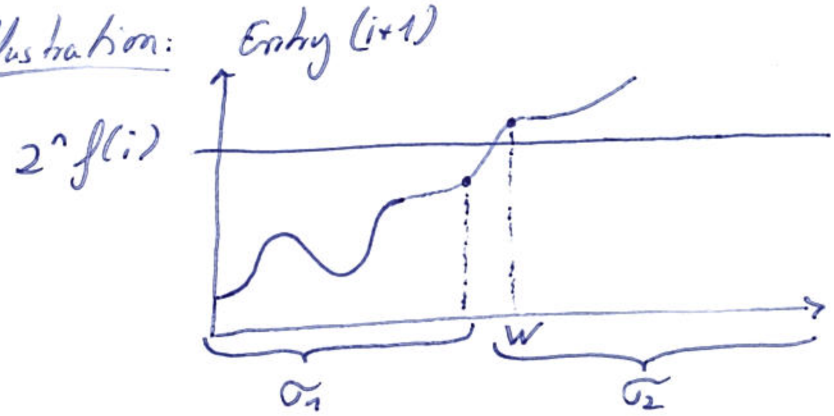
- σ_1 <u>is</u> $(i+1)$-$(2^\wedge f(i))$-bounded and
- σ_2 starts with a vector W that is <u>not</u> $(i+1)$-$(2^\wedge f(i))$-bounded.

Wlog. assume

$$W(i+1) \geq 2^\wedge f(i),$$

which means entry $(i+1)$ exceeds $2^\wedge f(i)$.

<u>Illustration:</u>



↳ Like in Case 1, we argue that

$$|\sigma_1| \leq (2^\wedge f(i))^{i+1}.$$

↳ Since σ_2 is an i-bounded, i-covering path in (N, W), there is an alternative path σ_2' that is also i-bounded and i-covering and moreover satisfies

$$|\sigma_2'| \leq f(i).$$

<u>Goal:</u> Show that also

$$\sigma_1 . \sigma_2'$$

is an $(i+1)$-bounded and $(i+1)$-covering path in (N, V).

Clearly, $\sigma_1 \sigma_2'$ has a length of
at most

$$\underbrace{(2^n f(i))^{i+1}}_{\sigma_1} + \underbrace{f(i)}_{\sigma_2'} .$$

↳ All edge weights in the PN are $\leq 2^n$
(by definition of vector size as
the length of the binary encoding
and the fact that $n = \mathrm{Size}(N, M_1, M_2)$).

↳ Since $|\sigma_2'| \leq f(i)$,
and since a path (of vectors) of length $f(i)$
has at most $f(i)-1$ transitions,
σ_2' can remove at most

$$2^n (f(i)-1)$$

tokens from $W(i+1)$.

↳ Since $W(i+1) \geq 2^n f(i)$,
this leaves us with

$$\geq 2^n f(i) - 2^n (f(i)-1) = 2^n$$

tokens.

↳ Again by the definition of vector sizes,
M_2 has at most 2^n tokens per place.

↳ This means

$$\sigma_1 . \sigma_2'$$

is $(i+1)$-bounded and $(i+1)$-covering. □

Goal: Give an upper bound on $f(k)$ that does not need recursion.

Approach: (1) Give a simpler recursive function $g(k)$ that upper-bounds $f(k)$.

(2) Give a non-recursive (closed-form) upper bound for $g(k)$.

Definition:

Define function $g: \mathbb{N} \to \mathbb{N}$ as follows:

$g(0) := 2^{3n}$ and $g(i+1) := (g(i))^{3n}$ f.a. $0 \leq i < k < n$.

Lemma:

For all $0 \leq i \leq k$ we have

$$f(i) \leq g(i).$$

Proof:

By induction along i.

Base case: $i=0$: $f(0) = 1 < 8 \leq 2^{3n} = g(0)$.

Induction step: Assume the claim holds for $0 \leq i < k$.

Consider

$$
\begin{aligned}
& f(i+1) \\
\text{(Lemma above)} \quad &\leq (2^n f(i))^{i+1} + f(i) \\
&= 2^{n(i+1)} \cdot f(i)^{i+1} + f(i) \\
\text{(Induction hypothesis)} \quad &\leq 2^{n(i+1)} \cdot g(i)^{i+1} + g(i) \\
\text{(Lemma below)} \quad &\leq g(i) \cdot g(i)^{i+1} + g(i) \\
&\leq 2g(i)\, g(i)^{i+1} \\
&\leq 2g(i)\, g(i)^n \\
&= 2g(i)^{n+1} \\
&\leq g(i)^{n+2} \leq g(i)^{3n} = g(i+1).
\end{aligned}
$$

□

Lemma:

$2^{n(i+1)} \leq g(i)$ f.a. $0 \leq i \leq k$.

Proof:

Base case: ($i=0$) $2^n \leq 2^{3n}$.

Induction step: Assume the inequality holds for $0 \leq i < k$. Then

$$
\begin{aligned}
& 2^{n((i+1)+1)} \\
&= 2^{n(i+1)} \cdot 2^n \\
\text{(Induction hypothesis)} &\leq g(i) \cdot 2^n \\
&\leq g(i) \cdot g(0) \\
&\leq g(i)^2 \\
&\leq g(i+1).
\end{aligned}
$$

$\square$

The closed-form solution for $g(k)$ is as follows.

Lemma:

(a) $g(k) \leq 2^{(3n)^n}$

(b) $(3n)^n \leq 2^{c\, n \log n}$, where c is independent of n.

Proof:

(a)

$$
\begin{aligned}
& g(k) \\
\text{(Definition)} &= \underbrace{\left(\ldots\left(\left(2^{(3n)}\right)^{(3n)}\right)^{(3n)} \ldots\right)^{(3n)}}_{(k+1) \text{ powers of } (3n).} \\
&= \left(\ldots\left(\left(2^{(3n)}\right)^{(3n)}\right)^{(3n)} \ldots\right)^{(3n)(3n)} \\
&= 2^{(3n)^{k+1}} \\
&\leq 2^{(3n)^n}.
\end{aligned}
$$

(b) $(3n)^n$

$= (3 \cdot 2^{\operatorname{ld} n})^n$

$\leq (2^2 \cdot 2^{\operatorname{ld} n})^n$

$= (2^{2+\operatorname{ld} n})^n$

$\leq (2^{4 \operatorname{ld} n})^n$

$= 2^{4n \operatorname{ld} n}$

$= 2^{(4l) n \log n}$. $\square$

<u>Altogether</u>:

There is a k-bounded, k-covering path in (N, V) of length $\leq 2^{2^{cn\log n}}$.

<u>Lemma</u>:

In all markings on this path, the token count is $\leq 2^{2^{dn\log n}}$.

<u>Proof</u>:

Note that every transition changes at most 2^n tokens:

$$\underbrace{2^n}_{\text{Initial marking}} + 2^n \underbrace{\left(2^{2^{cn\log n}} - 1\right)}_{\text{Transitions}}$$

$= 2^n \cdot 2^{2^{cn\log n}}$

$= 2^{2^{cn\log n} + n}$

$= 2^{2^{cn\log n} + 2^{l\log n}}$

(*) $\leq 2^{2^{s\cdot c\cdot n\log n} + 2^{t\cdot l\cdot \log n}}$

$\leq 2^{2^{s\cdot c\cdot n\log n} \cdot 2^{t\cdot l\cdot \log n}}$

$\leq 2^{2^{s\cdot c\cdot n\log n} \cdot 2^{t\cdot l\cdot n\log n}}$

$= 2^{2^{(sc+tl) n\log n}}$. $\square$

(*) s is chosen such that $2^{s\cdot c\cdot n\log n} \geq 2$ and similar for t.