Exercises to the lecture
Complexity Theory
Sheet 8

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Thomas Haasi Delivery until 14.02.2022 at 15:00

Exercise 8.1 (Modeling via Polynomials)

We use the polynomial approach to describe the solutions to the following problem:

k-PATH	
Input:	A directed graph $G = (V, E)$ and $k \in \mathbb{N}$.
Parameter:	k.
Question:	Is there a path of k vertices in G?

To come up with the right polynomial, we need some preliminaries: A *walk* is a sequence of k vertices $v_1, \ldots, v_k \in V$, with potential repetition. A *labeling* is a bijective function $\ell : [1..k] \to [1..k]$. A *labeled walk* is a pair (W, ℓ), where W is a walk and ℓ is a labeling. Intuitively, every (visit of a) vertex v in W gets a unique label from ℓ.
Our variables are $X = \{x_{u,v} \mid (u, v) \in E\}$ and $Y = \{y_{v,j} \mid v \in V, j \in [1..k]\}$. Let (W, ℓ) be a labeled walk with $W = v_1, \ldots, v_k$, then we define the monomial:

$$\text{mon}_{W,\ell}(X, Y) = \Pi_{i=1}^{k-1} x_{v_i,v_{i+1}} \cdot \Pi_{i=1}^{k} y_{v_i,\ell(i)}.$$

Note that the monomial models precisely the labeled walk in terms of variables.
The instance polynomial is given by the sum of all such monomials:

$$P(X, Y) = \sum_{\substack{\text{walk} \\ W=v_1,\ldots,v_k}} \sum_{\ell \text{ a labeling}} \text{mon}_{W,\ell}(X, Y).$$

We show that $P(X, Y)$ does not vanish modulo 2 if and only if G contains a k-path.

a) We define a function T between labeled walks. Let (W, ℓ) be a labeled walk with $W = v_1, \ldots, v_k$. If W is not a path then there is a smallest pair of numbers (a, b) such that $a < b$ and $v_a = v_b$. Define the labeling ℓ' as follows:

$$\ell'(x) = \begin{cases} \ell(b), x = a \\ \ell(a), x = b \\ \ell(x), \text{otherwise.} \end{cases}$$

The function T maps a labeled walk (W, ℓ) to the labeled walk (W, ℓ') if W is not a path. If W is a path, T is the identity map.

Prove that $\text{mon}_{W,\ell} = \text{mon}_{T(W,\ell)}$ for each labeled walk (W, ℓ).

b) Show that $P(X,Y)$ takes the following form modulo 2:

$$P(X,Y) = \sum_{\substack{\text{path} \\ W=v_1,\ldots,v_k}} \sum_{\ell \text{ a labeling}} \text{mon}_{W,\ell}(X,Y).$$

c) Conclude that $P(X,Y)$ does not vanish modulo 2 if and only if G contains a k-path.

Exercise 8.2 (Zeta and Möbius Transform)

Prove the following equivalences:

a) $\zeta = \sigma\mu\sigma$, and

b) $\mu = \sigma\zeta\sigma$.

Hint: Recall that $\sigma\sigma = id$.

Exercise 8.3 (Cover Product)

Let V be a finite set and $f, g : \mathcal{P}(V) \to \mathbb{Z}$ functions. We define the *cover product* of f and g to be the function $f *_c g : \mathcal{P}(V) \to \mathbb{Z}$ with

$$(f *_c g)(X) = \sum_{\substack{A,B \subseteq X \\ A \cup B = X}} f(A) \cdot g(B).$$

Show that $\zeta(f *_c g) = (\zeta f) \cdot (\zeta g)$

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