7. Translation

<u>Goal:</u> · Assume we showed $DTIME(t_1) \subseteq DTIME(t_2)$
We develop a technique to get out of this
further inclusions (among large complexity classes) for free.

· The technique can be used to derive
the general Savitch and Immerman-Szelepcsényi Theorems
from $NL \subseteq DSPACE(\log n^2)$
and $NL = co\text{-}NL$.

<u>Idea:</u> · Artificially extend the input by a fresh symbol #
↳ a technique called <u>padding</u>

· Padding does not make the language more complicated,
but it reduces the computational effort
in the sense that now the input is larger.

7.1 Padding and the Translation Theorems

<u>Definition:</u>
Let $L \subseteq \Sigma^*$ be a language
and $f: \mathbb{N} \to \mathbb{N}$ a function with $f(n) \geq n$ f.a. $n \in \mathbb{N}$.
Let $\# \notin \Sigma$ be a fresh symbol.

We define:
$$Pad_f(L) := \{ x \#^{f(|x|) - |x|} \mid x \in L \} \subseteq (\Sigma \cup \{\#\})^*.$$

<u>Note:</u> Padding turns every word in L of length n
into a word from $L\#^*$ of length $f(n)$.

Theorem (Translation for time):

Let f, g be functions with $f(n), g(n) \geq n$ f.a. $n \in \mathbb{N}$.

Let g be monotone and time constructible.

Given 1^n, let $1^{f(n)}$ be computable in time $g(f(n))$.

For $L \subseteq \Sigma^*$, we have

$$Pad_f(L) \in {}^{D}_{N}TIME(O(g)) \text{ iff } L \in {}^{D}_{N}TIME(O(g \circ f)).$$

Proof:

We do the proof for DTIME, NTIME is similar.

"$\Rightarrow$" Let $x \in \Sigma^*$ be an input

We check $x \in L$ in $DTIME(O(g(f(|x|))))$ as follows.

↳ Compute $y = x\#^{f(|x|)-|x|}$ in time $O(g(f(|x|)))$.

↳ Check $y \in Pad_f(L)$ in $O(g(|y|))$.

Works by the hypothesis.

Note that $g(|y|) = g(f(|x|))$.

↳ By definition of $Pad_f(L)$:

$y \in Pad_f(L)$ iff $x \in L$.

"$\Leftarrow$" Let $x \in (\Sigma \cup \{\#\})^*$ be an input.

We check in $DTIME(O(g(|x|)))$ whether $x \in Pad_f(L)$ as follows.

↳ Check in time $|x| \leq g(|x|)$ whether $x \in w\#^*$ for some $w \in \Sigma^*$.

Let $x = w\#^{|x|-|w|}$.

↳ Compute $1^{g(|x|)}$ in time $O(g(|x|))$.

This works as g is time constructible and the binary representation can be converted to unary in $O(g(|x|))$ steps.

↳ We now decide in time $g(|x|)$ whether

$$\boxed{|x| = f(|w|)} \quad \text{holds.}$$

To this end, we compute $1^{f(|w|)}$ in time $g(f(|w|))$.

If the machine wants to compute more than $g(|x|)$ steps, reject.

Why? Since g is monotonic, we have

$$g(f(|w|)) > g(|x|) \implies f(|w|) > |x|.$$

If we managed to compute $1^{f(|w|)}$, we can compare it to $1^{|x|}$.

If $|x| \neq f(|w|)$, reject.

Otherwise, $x = w\#^{f(|w|)-|w|}$.

↳ Check in time $O(g(f(|w|))) = O(g(|x|))$ whether $w \in L$.

□

Theorem (Translation for space):

Let $g(n) \geq \log n$ be space constructible.

Let $f(n) \geq n$ and so that given an input 1^n we can compute $\text{bin } f(n)$ in space $g(f(n))$.

For $L \subseteq \Sigma^*$, we have

$$Pad_f(L) \in {}^{D}_{N}SPACE(g) \quad \text{iff} \quad L \in {}^{D}_{N}SPACE(g \circ f).$$

7.2 Applications of the Translation Theorems

Consequence of the translation results:

↳ It is more likely that higher complexity classes will collapse.

↳ Phrased differently, to show a separation among complexity classes one should consider the lower end of the hierarchy.

Lemma (An implication of Kuroda I (next lecture)):

$DSPACE(n) \neq NSPACE(n) \Rightarrow L \neq NL$.

Proof:

We proceed by contraposition and assume $NL \subseteq L$.

From this we derive $NSPACE(n) \subseteq DSPACE(n)$.

Let $L \in NSPACE(n)$.

We note that $n = \log \circ \exp$ and

thus the translation theorem for space applies.

It yields

$$Pad_{exp}(L) \in NSPACE(O(\log n)) = NL.$$

By our assumption

$$NL \subseteq L = DSPACE(O(\log n)).$$

Thus, we can apply the translation for space again

and get

$$L \in DSPACE(O(\log \circ \exp)) = DSPACE(n).$$

The last equality uses tape compression. $\square$

Theorem:

$P \neq DSPACE(n)$, P is not the class of deterministic, context-sensitive languages.

Proof:

Consider a language $L \in DSPACE(n^2) \setminus DSPACE(n)$.

Such a language exists by the deterministic space hierarchy result.

Consider the padding function $f(n) = n^2$.

We get $Pad_f(L) \in DSPACE(n)$.

If now $DSPACE(n) = P$, we had

$$Pad_f(L) \in DTIME(O(n^k)) \quad \text{for some } k \in \mathbb{N}.$$

With the translation theorem for time,

$$L \in DTIME(O((n^2)^k)) = DTIME(O(n^{2k})).$$

But then

$$L \in P = DSPACE(n).$$

$\lightning$ This contradicts the choice of $L \notin DSPACE(n)$. $\square$

<u>Remark:</u>

Today $P \subsetneq DSPACE(n)$, $DSPACE(n) \subsetneq P$,
and $DSPACE(\log n) = P$ are still possible.