27. Advanced Topics in Parameterized Complexity

27.1 Bounded Treewidth

Goal: · Study properties of graphs.
· Assume the parameter indicates how tree-like a graph is.

Idea: · Use tree automata techniques
to accept the trees
that represent the graphs having the property of interest.

Technically: Courcelle's theorem

↳ Logic-based meta-theorem
for establishing that various graph-theoretic properties
are decidable in linear FPT,
when the input parameter is the treewidth.

↳ Needs Bodlaender's result:
Tree-decompositions of width k
can be computed in linear time (for every k there is a linear-time algorithm).

Formulation 1:
If the property φ of interest is expressible in MSO on graphs,
then, parameterizing by the treewidth of the input G,
one can determine in linear FPT whether $G \models \varphi$
(G has the property).

Formulation 2:
Given graph G and $\varphi \in MSO[\text{Graphs}]$, one can determine
in $f(tw(G) + |\varphi|) \cdot n^c$ for some c, whether $G \models \varphi$.

Examples: Hamiltonicity, k-coloring.

27.2 Well-Quasi Orderings

Goal: • Some sets have finite subsets / bases of minimal elements:

↳ polynomial ideals (Hilbert)

↳ graphs wrt. minors (Robertson & Seymore)

↳ trees wrt. contraction (Kruskal)

↳ words wrt. subsequences (Higman).

• Use this finiteness for algorithms.

WQO-Principle:

• Suppose $(S, \leq)$ is a wqo

• Assume that the following problem is polynomial time:

ABOVE(x):

Given: $y \in S$.

Parameter: $x \in S$.

Question: $x \leq y$?

• Then for any upward-closed set $\mathcal{U}$ of $(S, \leq)$ one can decide "$y \in \mathcal{U}$" in polynomial time.

Why?

$$\mathcal{U} = \min(\mathcal{U})\uparrow = \{x_1, \dots, x_n\}\uparrow = x_1\uparrow \cup \dots \cup x_n\uparrow.$$

Hence:

$$y \in \mathcal{U} \quad \text{iff} \quad x_1 \leq y \vee \dots \vee x_n \leq y.$$

Obstruction-Principle:

• Let $(S, \leq)$ be a wqo and D a downward-closed set of S.

• O is an obstruction set for D, if

$$x \in D \quad \text{iff} \quad \forall y \in O : y \not\leq x$$

- Suppose $ABOVE(x) \in P$ for all $x \in S$.

  Then "$y \in D$" is in P.

  The reason is that D has a finite obstruction set, by the wqo assumption

  (the complement of D is upward-closed, and hence has a finite basis).