Towards Algorithms

Problem: PARITY

Given: Parity game G with finite set of positions, position $p \in Pos$, and player $i \in \{A, P\}$.

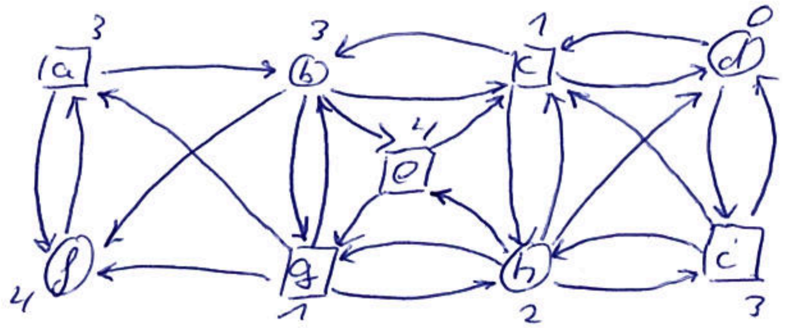
Question: Does $p \in W_i$ hold?

↳ First solve more general problem: compute W_A and W_P for a given parity game G

↳ Turn the above proof into recursive algorithm

The algorithm – on an example

Consider



- Let N be the set of positions with highest priority. Say this priority is even.

  In the example:

  $N = \{f, e\}$

- Compute $Attr_A(N)$.

  If A can ensure this attractor is visited infinitely often, she wins.

  In the example:

  $Attr_A(N) = \{a, b, e, f, g, h\}$

- Compute recursively positions X from which P wins $Pos \setminus Attr_A(N)$.

  In the example:

  $X = \{c, d, i\}$

Case 1: X is empty:

A wins everywhere

↳ A wins on $Pos \setminus Attr_A(N)$

↳ A certainly wins on $Attr_A(N)$.

Case 2: X is not empty

↳ From X, P wins the full game, not only $G|_{Pos \setminus Attr_A(N)}$

↳ Reason is that A cannot move to $Attr_A(N)$.

Compute $Y = Attr_P(X)$:

P wins on all positions in Y.

In the example:

$Y = \{c, d, i, e\}$.

Solve the remaining game $Pos \setminus Y$ by recursion.

Well-founded as $Y \neq \emptyset$.

Algorithm (McNZ Solve):

input: $G = (Pos_A, Pos_P, \to, \Omega)$.

begin:

$n := \max\{\Omega(p) \mid p \in Pos\}$

if $n = 0$ then

return $W_A = Pos$, $W_P = \emptyset$;

end if

if n = even then

$\sigma = A$, $\tau = P$;

else

$\sigma = P$, $\tau = A$;

end if.

$N := \{p \in Pos \mid \Omega(p) = n\}$;

$A_\sigma := Attr_\sigma(N)$;

$(W'_\sigma, W'_\tau) :=$ McNZSolve$(G|_{Pos \setminus A_\sigma})$;

if $W'_\tau = \emptyset$ then

return $(W_\sigma := W'_\sigma \cup A_\sigma, W_\tau := \emptyset)$

end if

$H_\tau := Attr_\tau(W_\tau')$;

$(W_\sigma'', W_\tau'') := McNZSolve(G|_{Pos \setminus H_\tau})$;

<u>return</u> $(W_\sigma := W_\sigma'', W_\tau := H_\tau \cup W_\tau'')$;

<u>end</u>

<u>Theorem:</u>

PARITY is in $NP \cap coNP$.

<u>Proof:</u>

Given game G, position p, player Π.

<u>In NP:</u>

↳ Guess a positional strategy s for player Π.

Can be done in $O(|\to|)$.

↳ Consider subgraph G' induced by s:

$\to' := \{(p'; s(p')) \mid p' \in Pos_\Pi\} \cup \to \cap Pos_P \times Pos$.

Can be constructed in $O(|\to|)$

↳ s is a winning strategy iff highest priority in every cycle that is reachable from p is even.

Can again be checked in polynomial time.

<u>In coNP:</u>

↳ Remember, PARITY in coNP iff complement in NP.

So given game G, position p, player Π, check if $p \notin W_\Pi$.

↳ Looks like we have to consider all strategies.

↳ But $p \notin W_\Pi$ iff $p \in W_P$ by <u>determinacy</u>.

↳ Use above NP algorithm to search for strategy for P. □