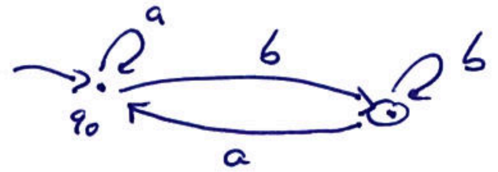
<u>Example</u> (Complementation of Büchi automata)

Consider

$\mathcal{A} =$



We characterize the transition equivalence classes $[u]_{\sim_\mathcal{A}}$ by two relations on states:

$$R_{[u]_{\sim_\mathcal{A}}} := \{ (q, q') \mid q \xrightarrow{u} q' \} \subseteq Q \times Q$$

$$R^{fin}_{[u]_{\sim_\mathcal{A}}} := \{ (q, q') \mid q \xrightarrow{u} q' \} \subseteq Q \times Q.$$

That these relations <u>characterize</u> the classes means

$$[u]_{\sim_\mathcal{A}} = [v]_{\sim_\mathcal{A}} \quad \text{iff} \quad R_{[u]_{\sim_\mathcal{A}}} = R_{[v]_{\sim_\mathcal{A}}} \text{ and } R^{fin}_{[u]_{\sim_\mathcal{A}}} = R^{fin}_{[v]_{\sim_\mathcal{A}}}.$$

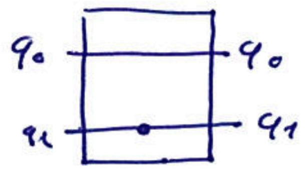
We typically represent both relations by one box:

$Box(u) := R_{[u]_{\sim_\mathcal{A}}}$ and $R^{fin}_{[u]_{\sim_\mathcal{A}}}$,

where "and" is represented by —•—.

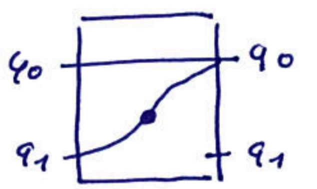
<u>In the example:</u>

- $Box(\varepsilon) =$



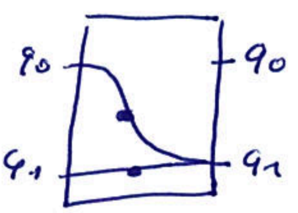
$= R_{[\varepsilon]_{\sim_\mathcal{A}}}$ and $R^{fin}_{[\varepsilon]_{\sim_\mathcal{A}}}$.

- $Box(a) =$



$= R_{[a]_{\sim_\mathcal{A}}}$ and $R^{fin}_{[a]_{\sim_\mathcal{A}}}$.

- $Box(b) =$



$= R_{[b]_{\sim_\mathcal{A}}}$ and $R^{fin}_{[b]_{\sim_\mathcal{A}}}$.

To compose the relations $R, S \subseteq Q \times Q$ on states, we define

$$R ; S := \{(q, q') \in Q \times Q \mid \exists q'' : (q, q'') \in R \text{ and } (q'', q') \in S\}.$$

(also denoted $S \circ R$)

One can check that

$$R_{[uv]_{\sim_{\mathcal{A}}}} = R_{[u]_{\sim_{\mathcal{A}}}} ; R_{[v]_{\sim_{\mathcal{A}}}} \text{ and}$$

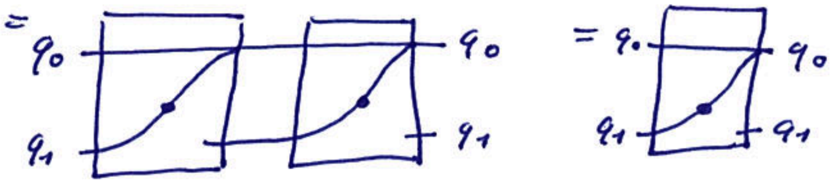
$$R^{fin}_{[uv]_{\sim_{\mathcal{A}}}} = (R^{fin}_{[u]_{\sim_{\mathcal{A}}}} ; R_{[v]_{\sim_{\mathcal{A}}}}) \cup (R_{[u]_{\sim_{\mathcal{A}}}} ; R^{fin}_{[v]_{\sim_{\mathcal{A}}}}).$$

If we represent the two relations by boxes, we simply glue them together:

$$Box(uv) = Box(u) ; Box(v)$$

<u>In the example:</u>

- $Box(aa) = Box(a) ; Box(a)$



$= Box(a).$

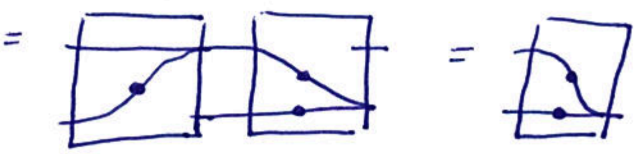
This means $R_{[aa]_{\sim_{\mathcal{A}}}} = R_{[a]_{\sim_{\mathcal{A}}}}$ and

$R^{fin}_{[aa]_{\sim_{\mathcal{A}}}} = R^{fin}_{[a]_{\sim_{\mathcal{A}}}}$.

Since $R_{[aa]_{\sim_{\mathcal{A}}}}$ and $R^{fin}_{[aa]_{\sim_{\mathcal{A}}}}$ characterize $[aa]_{\sim_{\mathcal{A}}}$, we conclude

$$[aa]_{\sim_{\mathcal{A}}} = [a]_{\sim_{\mathcal{A}}}.$$

- $Box(ab) = Box(a) ; Box(b)$



$= Box(b).$

With the idea of boxes, we can now construct all non-empty equivalence classes in $\sim_A$ by relational composition:

- ↳ Start with Box(a) ($R_{[a]_{\sim A}}$ and $R^{fin}_{[a]_{\sim A}}$) for all letters $a \in \Sigma$.
- ↳ Derive further classes $[uv]_{\sim A}$ by composition; of Box(u) and Box(v)
- ↳ Until a fixed point is reached, i.e., no more classes are found
  - ⇒ Such a fixed point is guaranteed to be found as there are only finitely many classes.

<u>In the example:</u>

;	Box(a)	Box(b)	Box(ba)
Box(a)	Box(a)	Box(b)	Box(ba)
Box(b)	Box(ba)	Box(b)	Box(ba)
Box(ba)	Box(ba)	Box(b)	Box(ba)

Note that several entries are algebraic consequences of others:

since $\text{Box}(aa) = \text{Box}(a)$,

we have
$$\begin{aligned}\text{Box}(ba); \text{Box}(a) &= (\text{Box}(b); \text{Box}(a)); \text{Box}(a)\\ &= \text{Box}(b); \text{Box}(aa)\\ &= \text{Box}(b); \text{Box}(a)\\ &= \text{Box}(ba).\end{aligned}$$

To compute $\mathcal{L}(\mathcal{A})$ and $\overline{\mathcal{L}(\mathcal{A})}$

it remains to check all compositions $[u]_{\sim_\mathcal{A}} . ([v]_{\sim_\mathcal{A}})^\omega$ for whether they do or do not belong to $\mathcal{L}(\mathcal{A})$:

1.) $[\varepsilon]_{\sim_\mathcal{A}} . ([a]_{\sim_\mathcal{A}})^\omega \cap \mathcal{L}(\mathcal{A}) = \emptyset$

<u>Why?</u>

$\varepsilon . a^\omega \notin \mathcal{L}(\mathcal{A})$.

2.) $[\varepsilon]_{\sim_\mathcal{A}} . ([b]_{\sim_\mathcal{A}})^\omega \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$,
indeed $[\varepsilon]_{\sim_\mathcal{A}} . ([b]_{\sim_\mathcal{A}})^\omega \subseteq \mathcal{L}(\mathcal{A})$

(note that always
↳ either $\cap = \emptyset$
↳ or $\subseteq$
(this was a lemma))

<u>Why?</u>

$\varepsilon\, b^\omega \in \mathcal{L}(\mathcal{A})$

...

16.) $[ba]_{\sim_\mathcal{A}} . ([ba]_{\sim_\mathcal{A}})^\omega \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$,
indeed $[ba]_{\sim_\mathcal{A}} . ([ba]_{\sim_\mathcal{A}})^\omega \subseteq \mathcal{L}(\mathcal{A})$.

<u>Why?</u>

$ba . (ba)^\omega \in \mathcal{L}(\mathcal{A})$.

In the end:

$$\mathcal{L}(\mathcal{A}) = \left([\varepsilon]_{\sim_\mathcal{A}} \cup [a]_{\sim_\mathcal{A}} \cup [b]_{\sim_\mathcal{A}} \cup [ba]_{\sim_\mathcal{A}}\right) . \left(([b]_{\sim_\mathcal{A}})^\omega \cup ([ba]_{\sim_\mathcal{A}})^\omega\right)$$

$$\overline{\mathcal{L}(\mathcal{A})} = \left([\varepsilon]_{\sim_\mathcal{A}} \cup [a]_{\sim_\mathcal{A}} \cup [b]_{\sim_\mathcal{A}} \cup [ba]_{\sim_\mathcal{A}}\right) . ([a]_{\sim_\mathcal{A}})^\omega.$$

To see that this covers Σ^ω, consider example words

- $\underbrace{ba}_{b}\underbrace{ba}_{b}\underbrace{ba}_{b}\ldots \in [\varepsilon]_{\sim_\mathcal{A}} . ([b]_{\sim_\mathcal{A}})^\omega \subseteq \mathcal{L}(\mathcal{A})$.
- $\underbrace{baba}_{ba}aaa\ldots \in [ba]_{\sim_\mathcal{A}} . ([a]_{\sim_\mathcal{A}})^\omega \cap \mathcal{L}(\mathcal{A}) = \emptyset$.