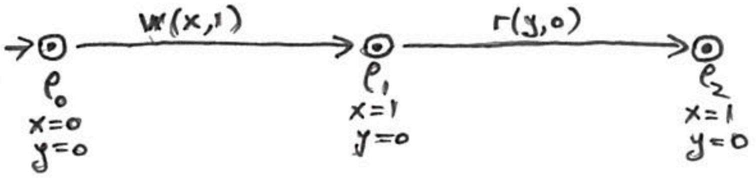
1.1. (a) Let $A_i = (Q_i, q_{i0}, \rightarrow_i, Q_{iF})$ for $i \in \{1,2\}$ and define the NFA $A_1 \cap A_2 := (Q_1 \times Q_2, \langle q_{10}, q_{20} \rangle, \rightarrow, Q_{1F} \times Q_{2F})$ where

$$\langle q_1, q_2 \rangle \xrightarrow{a} \langle q_1', q_2' \rangle \text{ iff. } q_1 \xrightarrow{a}_1 q_1' \text{ and } q_2 \xrightarrow{a}_2 q_2'.$$

(b) If $A_i = (Q_i, q_{i0}, \rightarrow_i, Q_{iF})$ for $i \in \{1,2\}$ then define the NFAs $A_i' := (Q_i, q_{i0}, \Rightarrow_i, Q_{iF})$ where

$$\Rightarrow_i \; := \; \rightarrow_i \cup \{ (q, a, q) \mid q \in Q_i \text{ and } a \in \Sigma_{3-i} \}.$$

1.2. (a) The states of A_{P_1} are elements of the set $L_1 \times (Var \longmapsto \mathbb{B})$ where L_1 is the set of labels of P_1.



The reachable part of A_{P_1} is depicted above. Given a state $(\ell, \nu) \in L_1 \times (Var \longmapsto \mathbb{B})$ one would interpret

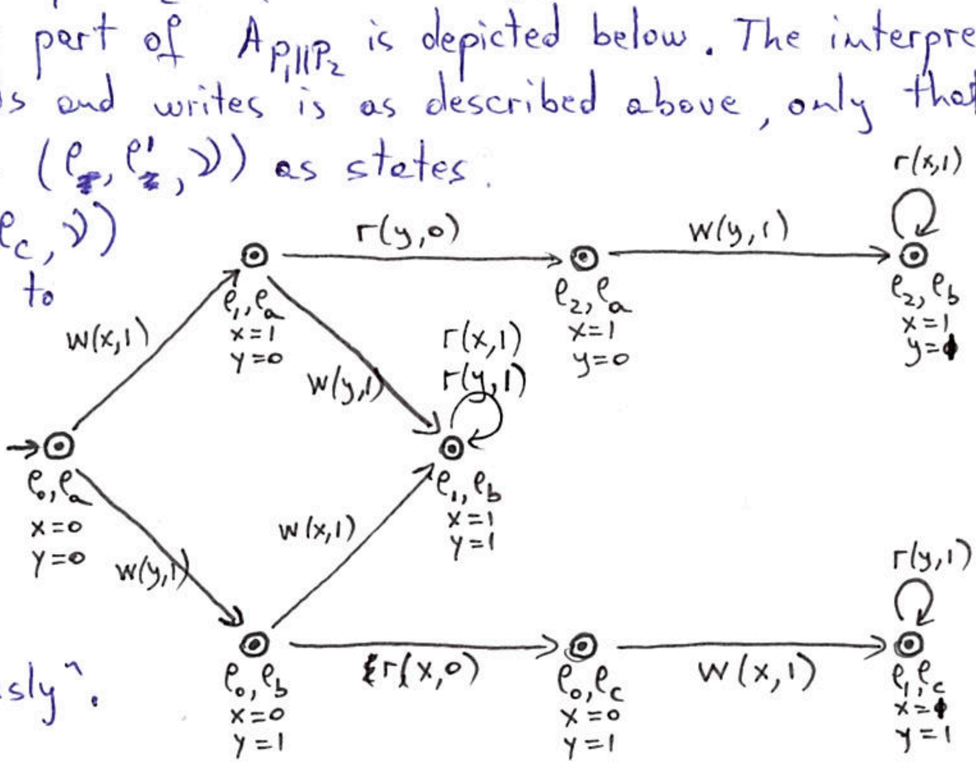
- ℓ : w(x,v) goto ℓ' as $(\ell, \nu) \xrightarrow{w(x,v)} (\ell', \nu[x := v])$
- ℓ : r(x,v) goto ℓ' as $\underbrace{(\ell, \nu) \xrightarrow{r(x,v)} (\ell', \nu)}_{\text{only when } \nu(x) = v\,!}$

(b) The states of $A_{P_1 \| P_2}$ are elements of

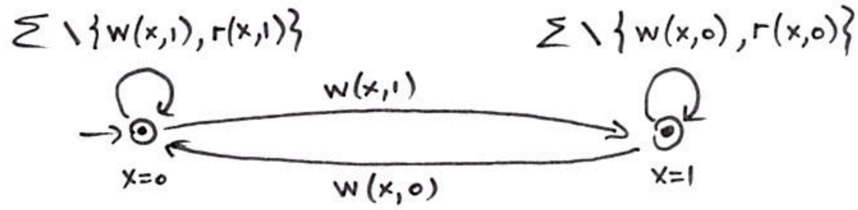
$$L_1 \times L_2 \times (Var \longmapsto \mathbb{B}).$$

The reachable part of $A_{P_1 \| P_2}$ is depicted below. The interpretation of reads and writes is as described above, only that one has triples (ℓ, ℓ', ν) as states.

Since (ℓ_2, ℓ_c, ν) does not belong to the reachable part of $A_{P_1 \| P_2}$ it means ℓ_2 and ℓ_c cannot be reached „simultaneously".



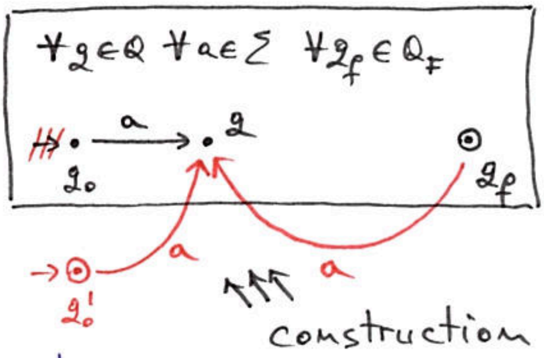
1.2. (c) Let $A_{Var} = A_x \cap A_y$ where A_x is depicted on the right.



Note that $\Sigma = \{ w(z,v) \mid z \in \{x,y\}, v \in \mathbb{B} \}$
$\cup \{ r(z,v) \mid z \in \{x,y\}, v \in \mathbb{B} \}$ in our example.

The conclusion one could draw is that reachability (e.g. of $\langle \ell_2, \ell_c, \vartheta \rangle$ for some $\vartheta \in Var \mapsto \mathbb{B}$) can be checked by analyzing the emptiness of $(L(C_{P_1}) \sqcup L(C_{P_2})) \cap L(A_{Var})$ where ℓ_2 and ℓ_c are the final states of C_{P_1} and C_{P_2}.

1.3. (a) We prove $L(A^*) = L(A)^*$ by double inclusion;



"$\underline{L(A^*) \subseteq L(A)^*}$". Let $w = a_1 r_1 \dots a_m r_m$ such that $a_i \in \Sigma$ and $r_i \in \Sigma^*$ be some word in $L(A^*)$.

Moreover, assume that the outlined transitions $a_i \in \Sigma$ are precisely the labels of red transitions taken to accept w, i.e. $\rho := q_0' \xrightarrow{a_1} q_1 \xrightarrow{r_1} q_{1f} \xrightarrow{a_2} q_2 \xrightarrow{r_2} q_{2f} \; --- \xrightarrow{a_m} q_m \xrightarrow{r_m} q_{mf}$ is an accepting run of w in A^*.

Then, by construction, $q_0 \xrightarrow{a_i r_i} q_{if}$ are all accepting runs in A. Hence, $a_i r_i \in L(A)$ and $w \in L(A)^m \subseteq L(A)^*$ ∎

"$\underline{L(A)^* \subseteq L(A^*)}$". We prove $L(A)^m \subseteq L(A^*)$ for all $m \in \mathbb{N}$ by induction.

Base case. $L(A)^0 = \{\varepsilon\} \subseteq L(A^*)$ since $\varepsilon \in L(A^*)$ by q_0'.

Step case. Let $w \in L(A)^k$ and $u \in L(A)$ for some $k \in \mathbb{N}$. By IH, $w \in L(A^*)$ so there exists an accepting run $q_0' \xrightarrow{w} q_f$ in A^* for some $q_f \in Q_F$.

Since $q_0 \xrightarrow{u} q_f'$ in A for some $q_f' \in Q_F$, by construction of A^* $q_0' \xrightarrow{w} q_f \xrightarrow{u} q_f'$ is a run of A^*.

Because $q_f' \in Q_F$ the run is accepting so $w.u \in L(A^*)$ and the induction is complete.

To conclude, $L(A)^* = \bigcup_{m=0}^{\infty} L(A)^m \subseteq L(A^*)$ ∎

1.3. (b) Let $A = (Q, q_0, \to, Q_F)$ and $A' = (\mathbb{P}(Q), \{q_0\}, \to', Q_F')$ its Rabin & Scott determinized automaton.

(i) For any $Q_1 \in \mathbb{P}(Q)$ and $a \in \Sigma$, $Q_1 \xrightarrow{a}' Q_2$ is defined by $Q_2 := \bigcup_{q \in Q_1} \{q' \in Q \mid q \xrightarrow{a} q'\}$. This proves Q_2 exists and is unique, hence the automaton A' is deterministic.

(ii) We must prove $L(A') = L(A)$.

„$\subseteq$" Let $a_1 \dots a_m \in L(A')$ i.e. $Q_0 \xrightarrow{a_1}' Q_1 \xrightarrow{a_2}' \dots \xrightarrow{a_m}' Q_m$ for $Q_0 = \{q_0\}$ and $Q_m \cap Q_F \neq \emptyset$. Take $q_m \in Q_m \cap Q_F$ arbitrary. By how $\to'$ is defined, for any $q_i \in Q_i$ there exists $q_{i-1} \in Q_{i-1}$ such that $q_{i-1} \xrightarrow{a_i} q_i$.

Therefore, for appropriately chosen q_i, $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_m} q_m$ is accepting in A, hence $a_1 \dots a_m \in L(A)$ ∎

„$\supseteq$" Let $a_1 \dots a_m \in L(A)$ i.e. $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_m} q_m$ and $q_m \in Q_F$. Then, for $Q_0 = \{q_0\}$ and $Q_i := \{q' \in Q \mid q \xrightarrow{a_i} q', q \in Q_{i-1}\}$ for $i \in \{1, \dots, m\}$ we get, by construction of A', that $Q_{i-1} \xrightarrow{a_i}' Q_i$ and $q_i \in Q_i$.

But then, $Q_0 \xrightarrow{a_1}' Q_1 \xrightarrow{a_2}' \dots \xrightarrow{a_m}' Q_m$ is an accepting run of A', hence $a_1 \dots a_m \in L(A')$ ∎

1.4. (a) If $L = U^*V$ then
$$\begin{aligned} UL \cup V &= UU^*V \cup \{\varepsilon\}V \\ &= (U^+ \cup U^0)V \\ &= U^*V = L \end{aligned}$$ ∎

(b) E.g. $U = \{\varepsilon, a\}$, $V = \{b\}$, $L = a^*(a+b) \neq U^*V$ satisfies $L = UL \cup V$ since
$$\begin{aligned} UL \cup V &= (\varepsilon + a)a^*(a+b) + b \\ &= a^*(a+b) + b \\ &= a^*(a+b) \\ &= L \end{aligned}$$

Alternatively to finding <u>an</u> example, one could prove that

$$U^*V' \text{ with } V' \supset V$$

are all solutions of $\begin{cases} L = UL \cup V \\ L \neq U^*V \end{cases}$.

1.4. (c) The language equations describing the given NFA are:

$$X_0 = aX_1 \cup bX_0 \cup \varepsilon \quad (1)$$
$$X_1 = aX_3 \cup bX_2 \quad (2)$$
$$X_2 = aX_1 \cup bX_0 \quad (3)$$
$$X_3 = aX_3 \cup bX_2 \quad (4)$$

Notice that $X_1 = X_3$ and $X_0 = X_2 \cup \varepsilon$ so $X_1 = aX_1 \cup bX_2 \quad (*)$

$$X_2 = aX_1 \cup bX_2 \cup b$$
$$= X_1 \cup b \quad (**)$$

From $(*)$ we have $X_1 = a^*bX_2$ by Arden's Lemma, which substituted in $(**)$ gives $X_2 = a^*bX_2 \cup b$.

Then by Arden's Lemma again $X_2 = (a^*b)^*b$, therefore $X_0 = (a^*b)^*b + \varepsilon$.

<u>Alternatively</u>, solving (4) by Arden gives $X_3 = a^*b\,X_2$, which substituted in (2) implies $X_1 = (a.a^*b + b)X_2 = a^*bX_2$.

By substituting X_1 in (1) and (3) we get

(5) $X_0 = a.a^*bX_2 \cup bX_0 \cup \varepsilon$

(6) $X_2 = a.a^*bX_2 \cup bX_0$,

From (6) by Arden we have $X_2 = (a^+b)^*bX_0$ which in (5) yields

$$X_0 = a^+b\,(a^+b)^*b\,X_0 \cup bX_0 \cup \varepsilon$$
$$= ((a^+b)^+b + b)\,X_0 \cup \varepsilon$$

hence, again by Arden's Lemma $X_0 = ((a^+b)^+b + b)^*$
$$= ((a^+b)^*b)^*.$$