Algorithmic Automata Theory
Exercise Sheet 2

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Exercise 1: Extensions of $WMSO$

a) Let us consider $WMSO[<, \text{suc}, .\,]$, the set of WMSO-formulas extended by concatenation. This means if φ, ψ are $WMSO[<, \text{suc}, .\,]$-formulas, then $\varphi.\psi$ is also a $WMSO[<, \text{suc}, .\,]$-formulas. Give semantics to concatenation (i.e. define when $\mathcal{S}(w), \mathcal{I} \models \varphi.\psi$ should be satisfied) so that

$$\mathcal{L}(\varphi.\psi) = \mathcal{L}(\varphi).\mathcal{L}(\psi) \ .$$

b) Present a $WMSO[<, \text{suc}]$ formula that is equivalent to $\varphi.\psi$.

c) For some fixed alphabet Σ, let us consider $WMSO[<, \text{suc}, [a]_{a\in\Sigma}]$, the set of WMSO-formulas extended by an operator $[a]$ for each symbol of the alphabet, i.e. if φ is a $WMSO[<, \text{suc}, [a]_{a\in\Sigma}]$-formula, then $[a]\varphi$ for any $a \in \Sigma$ is also a $WMSO[<, \text{suc}, [a]_{a\in\Sigma}]$-formula. Give semantics to $[a]\varphi$ so that

$$\mathcal{L}([a]\varphi) = \{w \in \Sigma^* \mid aw \in \mathcal{L}(\varphi)\}.$$

Furthermore, prove that $WMSO[<, \text{suc}, [a]_{a\in\Sigma}]$ is equally expressive as $WMSO[<, \text{suc}]$ by extending the translation form formulas to automata. This means that you should show how given an automaton for $\mathcal{L}(\varphi)$, one can construct an automaton for $\mathcal{L}([a]\varphi)$.

Exercise 2: Weak Dyadic Second Order Logic

Let $WDSO$ be like $WMSO$ with the modification that all second order variables X are dyadic instead of being monadic, i.e. they represent sets of pairs. Instead of having a predicate $X(x)$ (”x is in X”), we have predicate $X(x, y)$ (”(x, y) is in X”). The syntax and semantics of $WDSO$ are the same with those of $WMSO$ up to the predicate replacement:

$\mathcal{S}(w), \mathcal{I} \vDash X(x, y)$ iff. $(\mathcal{I}(x), \mathcal{I}(y)) \in \mathcal{I}(X)$
$\mathcal{S}(w), \mathcal{I} \vDash \exists X.\varphi$ iff. there is a finite set $M \subseteq D(w) \times D(w)$ such that $\mathcal{S}(w), \mathcal{I}[X \mapsto M] \vDash \varphi$.

We want to show that the class of languages representable by $WDSO$-formulas does strictly include than the set of regular languages.

a) Give (with arguments) a $WDSO$-formula that defines the language $\{a^n b^n \mid n \geq 0\}$.

b) Show how to translate a $WMSO$-formula ψ into a $WDSO$-formula that defines the same language.

Exercise 3: From $WMSO$ to Finite Automata

a) Using the method presented in the lecture, construct a finite automaton that accepts the language defined by the formula

$$\varphi = \exists x \exists y \colon x < y \wedge P_a(x) \wedge P_a(y) \ .$$

b) Present a $WMSO[<, \mathrm{suc}]$-formula that defines the language $((aa)^*b)^*$.

Exercise 4: $WMSO$ Expressiveness

a) Show that $WMSO[<, \mathrm{suc}]$ and $WMSO[\mathrm{suc}]$ are equally expressive.

b) Show that $WMSO[<, \mathrm{suc}]$ and $WMSO[<]$ are equally expressive.