Emanuele D'Osualdo
Sebastian Muskalla

Advanced Automata Theory
Exercise Sheet 4

TU Kaiserslautern
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Out: May 11 (**Updated version, May 12**) Due: **May 17, 10:00**

Because of the holiday on Monday, you can bring your submissions to the exercise class (Tuesday 10:00). If you submit (parts of your) solution on Friday, we can return them on Tuesday.

Exercise 1: Presburger formulas & Parikh images

a) Present a Presburger formula φ such that every bound variable occurs in **precisely** one atomic expression and such that

$$\mathrm{Sol}(\varphi) = \left\{ \begin{pmatrix} 2n+1 \\ n+3 \end{pmatrix} \middle| \ n \in \mathbb{N} \right\} \cup \left\{ \begin{pmatrix} 3n+1 \\ 2n+2 \end{pmatrix} \middle| \ n \in \mathbb{N} \right\} .$$

b) The **Parikh image** $\Psi : \Sigma^* \to \mathbb{N}^\Sigma$ mapps each word w to the vector $\Psi(w)$, where the components store the number of occurrences of each letter in w. For a language $\mathcal{L} \subseteq \Sigma^*$, let $\Psi(\mathcal{L}) = \{\Psi(w) \mid w \in \mathcal{L}\}$. For example for $\Sigma = \{a, b, c\}$:

$$\Psi(ababcb) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \text{ and } \Psi((aa)^*(bbb)^*) = \left\{ \begin{pmatrix} 2n \\ 3m \\ 0 \end{pmatrix} \middle| \ n, m \in \mathbb{N} \right\} .$$

Give an NFA A so that $\Psi(\mathcal{L}(A)) = \mathrm{Sol}(\varphi)$ for the Presburger formula φ from a).

Exercise 2: "Presburger $\Rightarrow$ NFA"-Algorithm

a) Prove the correctness of the construction given in class:
For every $q \in \mathbb{Z}$ and $w \in (\mathbb{B}^n)^*$, the automaton accepts w starting from q
iff w encodes $\vec{c}$ with $\vec{a}\,\vec{c} \le q$.

b) Construct a finite automaton over $\mathbb{B}$ for the atomic Presburger formula $x - 3y \le 1$.

Exercise 3: "Presburger $\Rightarrow$ NFA" for atomic formulas with equality

One can modify the algorithm for $\vec{a}\,\vec{x} \le b$ to produce an NFA for $\vec{a}\,\vec{x} = b$ by making the state $0 \in \mathbb{Z}$ the only accepting state and by changing the transition relation so that a transition

$$q \xrightarrow{\vec{\beta}} \frac{1}{2}(q - \vec{a}\,\vec{\beta})$$

is only added if $q - \vec{a}\,\vec{\beta}$ is even.

a) Use the modified algorithm to construct a finite automaton for $x - 2y = 1$.

b) Verify your result in a) by checking that

$$\mathcal{L}(A_{x-2y=1}) = \mathcal{L}(A_{x-2y\le 1}) \cap \mathcal{L}(A_{-x+2y\le -1}) .$$

Exercise 4: Semilinear sets

Let $c \in \mathbb{N}^n$ be a vector and let $P = \{p_0, \ldots, p_m\} \subseteq \mathbb{N}^n$ be a finite set of vectors. We define

$$\mathcal{L}(c, P) = \left\{ c + \sum_{i=0}^{m} k_i \cdot p_i \in \mathbb{N}^n \,\middle|\, k_1, ..., k_m \in \mathbb{N} \right\} .$$

A set is called **linear** if it is of the form $\mathcal{L}(c, P)$ for some $c \in \mathbb{N}^n$ and finite $P \subset \mathbb{N}^n$. A set is called **semi-linear** if it is a union of finitely many linear sets.

a) Prove that semi-linear sets are Presburger definable:
For any semi-linear set $S \subseteq \mathbb{N}^n$ there exists a Presburger formula φ_S such that $S = \mathrm{Sol}(\varphi_S)$.

b) A function $f\colon \mathbb{N}^n \to \mathbb{N}^m$ is linear if $f(x + y) = f(x) + f(y)$ and $f(k \cdot x) = k \cdot f(x)$ for all $k \in \mathbb{N}$.
Prove that semi-linear sets are closed under linear functions, i.e. if $S \subseteq \mathbb{N}^n$ is semi-linear and $f\colon \mathbb{N}^n \to \mathbb{N}^m$ is a linear function then $f(S) \subseteq \mathbb{N}^m$ is semi-linear.