Algebraic Automata Theory

Sheet 6, 2017-11-30

Exercise 1 [10 Points]

We introduced the category $\boldsymbol{uqnt}$ of unital quantales as the Eilenberg-Moore category of the composition of the list monad $\boldsymbol{L} = \langle (-)^*, \eta_{\boldsymbol{L}}, \mu_{\boldsymbol{L}} \rangle$ and the power-set monad $\boldsymbol{P} = \langle \mathbb{P}, \eta_{\boldsymbol{P}}, \mu_{\boldsymbol{P}} \rangle$ utilizing the familiar distributive law, or, equivalently, of the lifting of $\boldsymbol{P}$ to the Eilenberg-Moore-category of $\boldsymbol{set}^{\boldsymbol{L}} \cong \boldsymbol{mon}$ of $\boldsymbol{L}$. Hence we may see unital quantales as $\bigsqcup$-semilattices internal to $\boldsymbol{mon}$; morphisms are monoid-homomorphisms that preserve arbitrary suprema.

Show that unital quantales can also be seen as

- monoids internal to the category $\bigsqcup$-$\boldsymbol{slat}$ wrt. cartesian product, *i.e.*, $\bigsqcup$-semilattices equipped with an associative binary operation with neutral element;
- monoids enriched in the $\bigsqcup$-$\boldsymbol{slat}$, and hence 2-categories.

Show, moreover, that unital quantales are automatically closed as 2-categories, *i.e.*, have all residuations.

Remark: A more conventional point of view that avoids the notion of 2-cells is to consider $\bigsqcup$-$\boldsymbol{slat}$ as a *monoidal* category, *i.e.*, as a category $\mathscr{C}$ equipped with two functors $\mathscr{C} \times \mathscr{C} \xrightarrow{\otimes} \mathscr{C} \xleftarrow{I} \mathbf{1}$, subject to suitable laws expressing the associativity of $\otimes$ and the neutrality of I wrt. $\otimes$. Here "suitable" means that the laws do not have to be satisfied precisely with equality (often called "on the nose"), but only "up to coherent isomorphism". More precisely, besides $\otimes$ and I one has to specify natural isomorphisms with components $(A \otimes B) \otimes C \xrightarrow{\langle A, B, C \rangle \alpha} A \otimes (B \otimes C)$ and $I \otimes A \xrightarrow{A\lambda} A \xleftarrow{A\rho} A \otimes I$ that in turn have to satisfy certain axioms called *coherence conditions*.

Since the hom-sets of a $\bigsqcup$-semilattices are at most singletons, these coherence contitions are automatically satisfied in this particular situation.

Exercise 2 [6 Points]

From GAN (general abstrat nonsense) we know that the sinks underlying a colimit cocone are epi-sinks. Show or disprove: in the category $\boldsymbol{mon}$ of monoids and monoid-homomorphisms colimit cocones are jointly surjective.

Exercise 3 [14 Points]

Show or disprove that the class of recognizable subsets of monoids is closed under homomorphic images. More precisely: if M is a monoid and $L \subseteq M$ is recognizable, then for every homomorphism $M \xrightarrow{h} M'$ the direkt image $h[L] \subseteq M'$ is recognizable as well.

due on Thursday, 2017-12-07, 13:15,